To our Chemist
John Woods Beckm
with love from
his mother
Stockholm Louise Woods Beckm
Aug. 1928

from
J W Beckman
Grandson of
the translator

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Sylvester C Baker
from
Woods

AN

ELEMENTARY TREATISE

ON

STATICS,

By GASPARD MONGE.

With a Biographical Notice of the Author.

TRANSLATED FROM THE FRENCH,

By WOODS BAKER, A. M.

OF THE UNITED STATES COAST SURVEY.

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PREFACE

TO THE

AMERICAN EDITION.

A WANT has long been felt in this country of a good elementary treatise on Theoretical Mechanics. The books on that subject, in the English language, are mostly voluminous, and either adapted to the comprehension of those only who have mastered the various branches of analytical mathematics, or composed chiefly of practical and descriptive details. Hence, an accurate knowledge of the general laws, or grand fundamental truths of mechanics, so important to all men, in this eminently practical age and country, and especially to those who have some one of the useful arts as their daily occupation, has hitherto been attainable only by highly educated men.

One of the consequences of the want of a familiar acquaintance with the mechanical laws, upon which all machines, of whatever possible kind, must depend, is the large number of failures of inventions occurring every year. The authors of such attempts generally

have mental ingenuity enough, but, unfortunately, they have not the knowledge necessary to render their contrivances consistent with the laws of Nature, or adapted to attain the proposed ends by the best possible means. Machines, deficient in either of these essential matters, must sooner or later be discarded; and their disappointed inventors have then to regret the loss of their money and time, which proper information would have prevented.

To supply this urgent need, in part, the following little book upon Statics, or the science which treats of the equilibrium of forces applied to solid bodies, has been translated. It has long been known and highly admired by those who are familiar with the scientific literature of France; but to persons who have little or no acquaintance with French authors on such subjects, it may be well to mention, that correct information, so well digested, precise and clear, can be obtained from the literature of no other nation or language.

From the advertisement prefixed to the seventh French edition of this treatise, which has gone through eight editions in France alone, besides several that have been published at Brussels, the following extracts are made:

"The first edition of this work, which appeared in 1786, was specially intended for young candidates

for the Navy; now, it is one of the standard books most generally followed. A correct and clear style, rigorous demonstrations, and well connected propositions, have long caused it to be preferred for instruction in Statics. It is the first book in which there has been collected all that can be demonstrated in Statics synthetically. After having studied Euclid's Geometry, this work will be read without difficulty. Being modelled upon the method of the ancient geometers, it presents very clear ideas upon an abstract science, of which a great number of useful applications are made."

"A profound knowledge of Statics requires the aid of mathematical analysis, that is to say, of Algebra, and the Differential Calculus; but it is as important for beginners to study synthetical statics before analytical statics, as it is fit to precede the study of analytical geometry by that of elementary geometry."

"The discussion of the elementary machines offers to M. Monge an occasion for showing the truth of a principle, which the author of the *Mècanique Analytique* (Lagrange) has rendered so productive, and which is known as the *principle of virtual velocities.*"

The treatise on Statics, of Monge, is a necessary introduction to the work of Poisson, which is a large and thorough analytical treatise, composed for the pupils of the Polytechnic School of Paris.

The translator would here acknowledge his obligation to his friends, R. S. McCulloh, Esq., Professor of Natural Philosophy in the College of New Jersey, and J. B. Reynolds, Esq., Engineer and Lecturer on Mechanical Philosophy before the Franklin Institute, Philadelphia, for their kindness in giving him valuable advice and assistance in preparing this work for publication.

Washington, Nov. 1850.

BIOGRAPHICAL NOTICE

OF

GASPARD MONGE.

It may be neither uninteresting nor out of place here, to give a slight sketch of the life of the author of this work, who was one of the most distinguished of the eminent men of science in France at the close of the last century. It is selected from two papers,* written by his pupils, MM. Brisson and Dupin, and presented by M. Delambre in his *Analysis of the Labors of the Academy of Sciences during the year* 1818.

Gaspard Monge was born in 1746. . . . . . . His progress was so meritorious as to procure for him the Professorship of Physics in the College of Lyons, the year after he had completed his studies in that institution. . . . . . . Having gone to Beaune during vacation, he undertook to draw the plan of that city; for which purpose he was obliged to make the necessary instruments. He presented his labor to the rulers of his native city, who compensated the young author as generously as the limited means of the public purse would

* *Notice Historique sur Gaspard Monge*, by M. Brisson; and *Essai Historique sur les Services et les Travaux Scientifiques de G. Monge*, by M. Dupin. See the advertisement to the 7th edition of *Géometrie Descriptive, par G. Monge*, Paris, 1847.

permit. A lieutenant-colonel of military engineers, who was then at Beaune, obtained for Monge the position of draughtsman and pupil in the school for engineers, and conductors of works of fortification. . . . . . . As he designed with rare ability, his manual talent only was regarded; but he already felt his strength, and could not think, without indignation, of the exclusive esteem accorded to his mechanical skill. He said long afterwards: "I was tempted a thousand times to tear my drawings to pieces through spite at the ado which was made about them, as though I had no ability for producing any thing else." The director of the school caused him to make some practical calculations of a particular case of *defilement*, an operation serving to combine the relief and trace of a fortification, so that the defender may be sheltered from the fire of the assailant. Monge abandoned the mode previously followed, and discovered the first geometric and general method that has been given for this important operation. . . . . . . By applying successively his mathematical talent to different questions of an analogous kind, and generalizing his means of conceiving and operating, he at length succeeded in forming a body of doctrine, which was his *Descriptive Geometry*. . . . . . . For more than twenty years he found it impossible to get the application of his geometry to draughts of carpentry taught to the corps at Mezieres. He was more fortunate, however, in the application to stone cutting; he carefully followed the methods employed in this art, and improved them by rendering them more simple by means of his geometry.

His scientific labors secured him the nomination of Lecturer on Mathematics and Physics, to succeed Nollet and Bossut. He was afterwards appointed the regular professor, when, turning his attention to a series of natural phenomena, he made numerous experiments upon electricity, explained the phenomena relating to capillarity, and was the founder of an

ingenious system of meteorology. He effected the decomposition of water, and arrived at this great discovery without having had any knowledge of the researches made a short time before by Lavoisier, Laplace, and Cavendish. He was not satisfied with explaining the theories of science and their application to the students in the lecture rooms; but liked to conduct his pupils wherever the phenomena of nature and the works of art could render these applications interesting and apparent to the senses. He infused into his pupils his own ardor and enthusiasm, and rendered delightful, observations and investigations, which, had they been viewed only abstractly in the narrow precincts of a lecture-room, would have appeared a wearisome study.

In 1780, to attract Monge to Paris, he was associated with Bossut as Professor of the course of Hydrodynamics, established by Turgot. In order to reconcile the duties of the two places he had to fulfil, he passed six months of the year at Mezieres, and six months at Paris. The same year he was elected a member of the Academy of Sciences; and, at the death of Bezout, in 1783, was chosen to replace that celebrated examiner for the Navy. The Marquis de Castries frequently requested Monge to re-write the Elementary Course of Mathematics for the pupils of the Navy; but he always declined. "Bezout has left," said Monge, "a widow who has no other fortune than the writings of her husband, and I cannot think of snatching away the bread from the wife of a man who has rendered such important service to science and the country." The only elementary work which Monge published, was his *Treatise on Statics;* and, with the exception of a few passages, where the evidence adduced might be somewhat more rigorous, the Statics of Monge is a model of logic, clearness, and simplicity.

At an epoch when the public distress called into the higher ranks all useful and courageous talent to the succor of the

country, which was threatened with an invasion, Monge was appointed Minister of the Navy. He did every thing to retain in France, men eminent for their merit or bravery; and descended even to petition, to obtain the continuation of the services of Borda, in which he had the good fortune to succeed. He was one of the most active men in scientific labors for the safety of the state. To him is due the construction of the new grinding apparatus in the powder works of Grenelle, and the boring machines on the boats of the Seine. He spent the daytime in giving instructions and directions in the workshops, and the nights in preparing his *Treatise on the Art of making Cannons;* a work intended as a manual for the directors of foundries, and for artizans.

It was in his course of lectures at the Normal School that he delivered, for the first time, his Lessons on Descriptive Geometry, the secrets of which he had not been able sooner to reveal. Another institution, which preceded the Normal School in the order of conception, but which, being longer in maturing by its authors, followed it in the order of execution, realized in part the hopes that had been conceived in vain of founding the first Universal School opened in France. Monge, by uniting the result of long experience at Mezieres with his original and profound views, arranged the plan of studies, indicated their connection, and proposed practicable means of execution. Out of four hundred pupils, who had first entered the Polytechnic School, fifty of the best scholars were taken to form a preparatory school; and it was Monge almost alone, who trained them. He remained the whole day amongst them, giving them by turns lessons in Geometry and Analysis; exhorting, encouraging, and inflaming them, by that ardor, benevolence, and impetuosity of genius, which caused him to exhibit to his pupils the truths of science with irresistible force and charm. In the evening, Monge commenced his labors anew: he wrote the pages of Analysis which were to

serve for the text of his next lessons; and on the following day he was with his pupils at the first moment of their assembling.

The amiability of Monge was neither the premeditation of the sage, nor even the effect of education: it was an unaffected gentleness of disposition, which he owed to his happy organization. He was born to love and admire. He was excessive both in his admiration and his love; hence he did not always, perhaps, remain within the limits where passionless and cold reason would have restrained him. . . . . . . As he was the father of the pupils in the bosom of the school, he was likewise the friend of the soldier in camp.

While travelling through Italy, collecting the statues and paintings ceded to France by treaty, Monge had been struck by the strange contrast which the monuments of the Greeks presented to those of the Egyptians, transported to the banks of the Tiber by Augustus and his successors. The comparative character of the antique monuments became the subject of frequent conversation between the conqueror of Italy and the commissioner, who collected for his country the most beautiful of the fruits of victory. Monge conceived the idea of extending the domain of history beyond the fabulous ages of Greece; of ascertaining, with the certainty of geometry, what the works of the ancient sages of the East were; of recovering, by contemplating their monuments, that which had been—the processes of their arts, the customs of their public life, the order and majesty of their feasts and ceremonies.

Charged by the General-in-chief, Napoleon, with bringing the treaty of Campo Formio to the Directory, he was, shortly afterwards, in the first rank of those who composed the Commission of Science and Art, which accompanied the Egyptian expedition. He was the first President of the Institute of Egypt, formed upon the model of the Institute of France. He

visited the Pyramids twice, saw the obelisk and the great walls of Heliopolis, and studied the remains of antiquity scattered around Cairo and Alexandria. During a wearisome march in the interior of the desert, he discovered (as he supposed) the cause of that wonderful phenomenon known as the *mirage.* At the time of the revolt at Cairo, there were only a few detachments of troops in the city, and the palace of the Institute was guarded only by the scientific corps. It was proposed that they should make their way, sword in hand, to the headquarters; but Monge and Berthollet, considering that the palace contained books, manuscripts, plans, and antiquities, the fruits of the expedition, maintained that the preservation of this precious deposit was the first duty of the scientific corps; and they decided to die, if necessary, in defending this treasure, rather than to desert it.

Monge presided over the Commission of Science and Art of Egypt; and by his counsels, contributed greatly to the judicious conception of the plan, its arrangement, the proportion of the principal parts, and the means of improving the arts of execution.

He had an inimitable manner of expounding the most abstract truths, and of rendering them clear by the language of action. . . . . . . It was, however, only by combating nature that he became so excellent a professor: for he spoke with difficulty, indeed almost stammered; his utterance, causing him to drawl some syllables and utter others with too great rapidity. His physiognomy, habitually calm, presented an aspect of meditation; but when he spoke, he appeared suddenly as though a different man: a new fire instantly lighted up his eyes, his features became animated, and his figure inspired. . . . . . .

Enfeebled by age, Monge became the victim of an imagination, which, according as the times were adverse or propitious, carried him beyond well founded fears or hopes. . . . . . .

The regulations of the service did not permit generous youth, at his funeral, to deposite the palm of gratitude and regret upon the tomb of their first benefactor; but, with the early dawn which followed the day of his obsequies, the pupils silently wended their way to his place of sepulture, and deposited there a branch of oak, to which they suspended a laurel crown. Twenty-three of the former pupils of the Polytechnic School, all residents of the city of Douai, united spontaneously, and decided to write in common to M. Berthollet, begging him to superintend the erection of a monument to be raised at the expense of the old pupils of the Polytechnic School, in honor of Gaspard Monge.

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AN

ELEMENTARY TREATISE

ON

STATICS.

DEFINITIONS.

Every thing that is capable of affecting our senses is called a *body*, or *material substance.*

Bodies are divided into *solids* and *fluids.* A body is *solid* when the molecules which compose it are cohesive, and cannot be displaced among each other without effort: the metals, stones, wood, &c., are of this number. It is *fluid* when, on the contrary, all its molecules can be separated with the greatest ease; such are water, air, &c.

All bodies are *moveable,* that is, they may be transported from one place to another. A body is said to bo *at rest,* when all the parts which compose it remain each in the same place; and it is said to be in motion, when it changes place, or when the parts of which it is composed pass from one place to another.

A body at rest cannot enter into motion, and when in motion, cannot change the manner in which it moves,

without the action of some cause, to which has been given in general the name of *force or power.*

There are to be considered in a force, 1st, its *intensity*, that is to say, the effort which it makes to move the body, or the particle of the body to which it is applied; 2d, its *direction*, or the straight line in which it tends to move the point of the body upon which it acts.

When several forces are applied to the same body, two cases may happen: either these forces counterbalance and mutually destroy each other, when they are said to be *in equilibrium;* or, by reason of the action of all these forces, the body enters into motion.

Hence, the term *Mechanics* is given to that science whose object it is to find the effect which the application of determined forces must produce upon a body. This science is divided into two parts: the first considers the relations which the forces should have, in intensity and direction, so as to be in equilibrium, and is called *Statics;* the second, to which the name *Dynamics* has been given, determines the manner in which the body moves, when these forces do not entirely destroy each other.

Each of these parts is again divided into two others, according as the body, to which the forces are supposed to be applied, is solid or fluid. The part of Statics which treats of the equilibrium of forces applied to solid bodies, is named simply Statics, or *Statics proper;* and *Hydrostatics* is that which has for its object the equilibrium of forces applied to the different molecules of a fluid body.

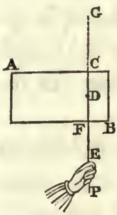
In this Treatise we will consider only the first of these two parts, that is, Statics proper.

CHAPTER FIRST.

OF THE COMPOSITION AND DECOMPOSITION OF FORCES.

1. *When a force* P, *applied to a determined point* C *of a solid body* AB, *draws or pushes this body in any direction* CF, *we may consider this force as though it were immediately applied to any other point* D *of the body, taken upon the direction of this force.*

Fig. 1.

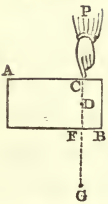


For all the points of the body, which are in the straight line CF, can neither approach towards, nor remove from, each other; and none of them can move along this line without causing all the others to move in the same manner as though the force were immediately applied to them.

We are likewise permitted to consider the force P *as though it were applied to any other point* G, *taken beyond the body, upon its direction, provided this point is invariably attached to the body.*

2. Hence it follows that if, upon the direction of the force P, there is found a fixed point D within the body, or an immovable obstacle beyond it, provided in the latter case the obstacle is invariably attached to the body, the force will be destroyed, and the body will remain at rest; for this force may be regarded as immediately applied to the fixed point, and its effect will be destroyed by the resistance of this point.

Fig. 2.



3. Reciprocally, if the force P, applied to the body AB, is destroyed by the resistance of a single fixed point, this point is found upon the direction of the force: for this point can destroy the effect of the force only by opposing the motion of the point of application C; and it cannot prevent this motion, unless it is upon the straight line which the force tends to make the point of application traverse.

4. *A point cannot move in several directions at the same time.*

5. When several forces, differently directed, are applied at the same time to the same point, either this point will remain at rest, or it will move in a single direction, and consequently in the same manner as though it were pushed or drawn by a single force along this direction and capable of the same effect.

6. Thus, whatever may be the number and directions of the forces applied at the same time to the same point, there always exists a single force which can move it, or tends to move it in the same manner as all these forces together; this single force is named the *resultant* of the former; and they, with reference to the resultant, are named the *component forces.*

The operation, by which we seek the resultant of several given component forces, is named the *composition of forces;* and that by which we find the components, when the resultant is known, is named the *decomposition of forces.*

7. *Two forces are equal, when, being applied to the same point and directly opposed, they destroy each other and produce an equilibrium.*

Reciprocally, when two forces are in equilibrium, they are equal and directly opposed.

8. For if several forces, differently directed, be applied to the same point, it is necessary, in order to put them in equilibrium, or to destroy the effect of their resultant, that a single force, equal to this resultant, should be applied to this point and directly opposed to it, or that several forces be applied, the resultant of which is equal and directly opposed to the resultant of the former.

9. Reciprocally, when several forces, differently directed and applied to the same point, are in equilibrium, their resultant is zero; or what is the same thing, any one of these forces is equal and directly opposed to the resultant of all the others; or, lastly, the resultant of any number of these forces is equal and directly opposed to the resultant of all the others.

10. *The resultant of two forces, applied to the same point, is in the plane determined by the directions of these forces; and it is necessarily included in the angle formed by these two forces.*

If the resultant were not in the plane of the two forces, there would be no reason why it should be above the plane rather than below: it cannot be at the same time in two different positions; hence it is in the plane of the two forces; moreover, it is included in the angle of the lines along which these forces are directed; for there is no force tending to move the point into the space adjacent to this angle; hence it will remain in the angle itself.

11. *A force is the multiple of another force when*

it is formed by the union of several forces equal to the latter.

Thus, if we apply to the same point and in the same direction several forces equal to each other, and if we take any one of these forces as unity, the multiple force will be expressed by a number equal to that of the added forces.

As it is always possible to compare numbers with lines, we may represent a force by a straight line taken upon its direction, and its multiple force by another straight line, a multiple of the former; in Statics, frequent use is made of this geometrical construction.

12. *Three equal forces, which are applied to the same point, and which divide the circumference, of which this point is the centre, into three equal parts, are necessarily in equilibrium.*

The point to which these three forces are applied can move only in a single direction; but the line along which it would move, could be placed in three or six manners perfectly similar with reference to the three forces; now there is no reason that the motion of the point should take place in one direction rather than in another; hence it will remain at rest.

The application of these three equal forces to the same point proves, that there is evidently for this case a resultant of these forces, since any one of the three forces makes an equilibrium with the two others, and it is evident that the direction of one of these forces divides the angle formed by the two others, into two equal parts; it is not less evident that *the resultant of any two equal forces meeting in a point, divides the angle formed by these two forces into two equal parts.*

13. *If several forces are applied to the same point along the same line and act in the same direction, it follows, from proposition No.* 11, *that their resultant is a single force, equal to their sum, acting along the same line, and in the same direction.*

Hence, to make an equilibrium of all these forces, it is necessary to apply to the same point and in the opposite direction, a force equal to their sum; for this force will be equal and directly opposed to their resultant.

14. From this it follows: 1st. If two unequal forces are applied to the same point in opposite directions, their resultant is in the direction of the greater, and is equal to their difference: for the greater of these two forces may be regarded as composed of two others having the same direction; one of which is equal to the smaller force, and the other equal to the difference: now the first of these two latter forces is destroyed by the smaller (7); then to move the point, there remains only the difference, which is in the direction of the greater.

2d. If any number of forces are applied to the same point, some being placed in one direction, and the others directly opposed, after having taken the sum of all those acting in one of the two directions, and the sum of all those acting in the contrary direction, the resultant of all these forces is equal to the difference of the two sums (12), and is in the direction of the greater.

Hence, to produce an equilibrium of all these forces, it is necessary to apply to the same point, and along the direction of the smaller of the two sums, a force equal to the difference of these sums; for this force will be equal and directly opposed to their resultant.

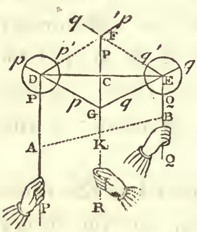
THEOREM.

15. *If at the extremities of an inflexible straight line* AB, *two equal forces* P, Q, *be applied along the lines* AP, BQ, *parallel to each other, and which act in the same direction:*

1*st. The direction of the resultant* R *of these two forces is parallel to the straight lines* AP, BQ, *and passes through the middle of* AB ;

2*d. This resultant is equal to the sum* P+Q *of the two forces.*

Fig. 3.



Demonstration.—Let another inflexible straight line DE be drawn perpendicular to the directions of the two forces P and Q, and invariably connected with the straight line AB: having prolonged the directions of the two forces, we may suppose that these two forces act (1) at the points D and E; moreover, we may apply to the same points, the forces p, p', and q, q', equal to the forces P and Q, so that the three equal forces P, p, p', meeting at the point D, will divide the circumference, which has its centre at the point D, into three equal parts, and so that the three forces Q, q, q', equal to each other and to the first, will divide in like manner the circumference which has its centre at the point E, into three equal parts.

We have seen (12) that the force P is equal and opposed to the resultant of the couple p, p'; that the

force Q is equal and opposed to the resultant of the couple q, q'; hence the forces P and Q have a resultant equal and opposed to that of the system of the two couples q, q' and p, p'; the two forces p' q', applied to the point F of their direction, have for resultant a third force equal to each of the first two, and acting in the direction CF; likewise the two forces p and q, applied to the point G in the line of their direction, have for resultant a third force equal to each of them, and directed along the line GC; hence, the resultant of the four forces p, p', q, q' is a single force equal to two of the others, and acting along the line GCF, which divides the line DE into two equal parts; hence the resultant of the two equal forces P and Q is equal to the sum P+Q, and divides the line DE, or the line AB, to which these forces are applied, into two equal parts. (See a second demonstration of this theorem, No. 19.)

Corollary I.

16. Hence, to bring the two forces P and Q into an equilibrium, it is necessary to apply to the middle K of the straight line AB a third force equal to their sum, which will act in a contrary direction, and which direction will be parallel to the two lines AP, BQ; for this third force will be equal and directly opposed to their resultant.

Corollary II.

17. If an inflexible straight line be divided into any number of equal parts, and we apply to all the points

of division, equal and parallel forces, the resultant of all these forces will pass through the middle of the line, in a direction parallel to that of the forces, and will be equal to their sum.

For, all the partial resultants of these forces, considered two and two, and taken at equal distances from the middle point of the line, will pass through this point (15), in the direction of the forces, and each of them will be equal to the sum of the two forces which compose it; hence (13) the general resultant will also pass through the middle of the line, in the same direction, and will be equal to the sum of all the partial resultants, that is to say, to the sum of all the component forces.

THEOREM.

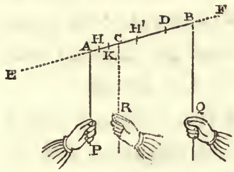
18. *If at the extremities of an inflexible straight line, two unequal forces* P *and* Q *be applied, whose lines of direction* AP, BQ, *are parallel to each other, and act in the same direction:*

1*st. The resultant* R *of these two forces is equal to their sum, and its direction is parallel to that of these forces;*

2*d. The point of application* C *of the resultant divides the line* AB *into two parts reciprocally proportional to the two forces, so that we have*

$$P : Q :: BC : AC.$$

Fig. 4.



DEMONSTRATION.—Suppose, in the first place, the two forces P and Q are commensurable; divide the line AB into two parts AD, DB, proportional to the forces P, Q; laying off the line AD from A to E, and the line BD from B to F, the line EF will be double AB; and since

$$P : Q :: AD : DB,$$

we shall have also,

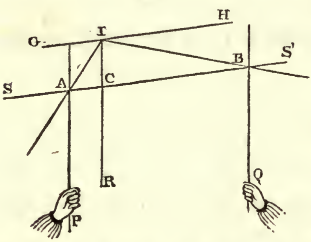
$$P : Q :: ED : DF.$$

Dividing the lines AD, DB into as many parts as there are units in the forces P and Q, and repeating this division upon the lines AE, BF, the whole line EF will be divided into twice as many equal parts as there are units in the sum of the two forces P and Q; now, the middle of each of these divisions may be considered as the points of application of forces equal to each other and each equal to $\frac{P+Q}{2m}$ (17), m being the number of units of the sum P+Q; hence the resultant of all these forces will be also the resultant of the two forces P and Q; but the resultant of equal forces, distributed on each side of the middle of equal divisions of the same line, is equal to the sum of these forces, passes through the middle of this line, and acts in the same direction as the components: hence the resultant of the two forces P and Q is equal to the sum P+Q, passes through the middle point C of the line EF, and acts in the direction of the forces P and Q; moreover, from the construction of *Figure* 4, $CF = \frac{1}{2}EF = AB$; then subtracting the common part CB, we have $AC = BF = BD$. For the same reason

EC$=\frac{1}{2}$EF$=$AB; hence CB$=$AE$=$AD; hence the middle point C of EF is such that we have P : Q :: CB : CA; hence this point of application divides the line AB into two parts reciprocally proportional to the forces.

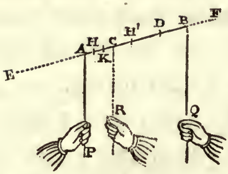
19. Let us suppose now that the two forces P and Q are incommensurable. Apply to the points A and B, and along the line AB, two equal and opposed forces S, S′; the resultant of the two forces P and Q will be the same as the resultant of the four forces P, S, Q, S′; the couple P, S has for resultant a line AI included in the angle SAP; the couple Q, S′ has for resultant a line directed along BI included in the angle QBS′; supposing these two resultants to be applied to the point of intersection I of their directions, and drawing through the point I a line GH parallel to AB, they will each be decomposed into two forces along the directions IH and GH; the forces in the direction GH being equal to S and S′ and opposed, they will destroy each other; the forces in the direction IR combine together and are equal to P$+$Q; hence, whether the two parallel and unequal forces P and Q be commensurable or incommensurable their resultant is parallel to their direction and equal to their sum.*

Fig. 5.



* This demonstration is also true when the two forces P and Q are equal to each other; consequently it is applicable to the theorem of No. 15.

Fig. 4.



20. The resultant R of the two forces P and Q, supposed to be incommensurable, passes through a point C in such a manner that we have

$$P : Q :: CB : CA.$$

For if it passed through another point H, situated between A and C, we should find a force Q′ applied in the direction Q, which would be such that the resultant of the commensurable forces P and Q′ would pass through a point K situated between the points C and H; this force Q′ would be the fourth term of the proportion

$$KB : KA :: P : Q',$$

the lines KB, KA, having for unit of measure a line smaller than CH; but we have $\frac{KB}{KA} > \frac{CB}{CA}$: hence the ratio $\frac{P}{Q'}$ is greater than the ratio $\frac{P}{Q}$. Hence Q′ is smaller than Q: consequently, the resultant of the two forces P and Q will necessarily pass between the points K and Q′; hence it is absurd to suppose that it passes through the point H situated beyond the point K. It might be demonstrated in the same manner that it cannot pass through a point H′ situated between C and B; hence it passes necessarily through the point C.

Corollary I.

21. Hence, in order that equilibrium may ensue between the two forces P, Q, it is necessary to divide the line AB at the point C into two parts reciprocally proportional to these two forces, and apply to the point C

a third force, equal to the sum P+Q, parallel to them and acting in the opposite direction.

Remark.

22. If the ratio of the forces P, Q, and the length of the line AB, were given in numbers, and it were desired to find the distances of the point C from the points A, B, the proportion

$$P : Q :: BC : AC$$

could not be employed directly, because in this proportion we would know only the first two terms; but it is easy to deduce from it the following,

$$P+Q : Q :: BC+AC : AC,$$

which, since BC+AC is equal to AB, becomes

$$P+Q : Q :: AB : AC,$$

in which the first three terms are known.

The distance BC can be found by the proportion

$$P+Q : P :: AB : BC,$$

which is likewise deduced from the first.

Corollary II.

23. When a single force R is applied to a point C of an inflexible straight line, it may always be decomposed into two others P, Q, which, being applied to the two points A, B, given upon the same line, and being directed parallel to RC, produce the same effect; and the intensity of the two forces is found by dividing the force R into two parts reciprocally proportional to the lines AC, CB, by means of the two following proportions:

$$AB : BC :: R : P,$$

$$AB : AC :: R : Q,$$

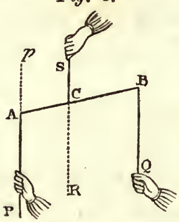
in each of which the first three terms are known; for

the resultant of the two forces P, Q, has the same intensity, is parallel to, and acts in the same direction as, the force R.

Corollary III.

24. Since *Fig.* 6 is the same in all respects as the preceding, if we apply to the point C of the line AB a

Fig. 6.



force S, equal and directly opposed to the resultant of the two forces P, Q, in such manner that we have

$$S=R=P+Q \ (19),$$

the three forces P, Q, S, will be in equilibrium, and each of the two forces P, Q, may be regarded as equal and directly opposed to the resultant of the two others. Hence the resultant of the two forces S, Q, which are parallel, and act in contrary directions, is a force *p* equal and directly opposed to the force P. Now, the force P is equal to the difference of the forces S, Q, and acts in the contrary direction to the greater, S, of these two forces: hence, 1st, the resultant *p* or —P of the two forces S, Q, is equal to their difference S—Q, and acts in the direction of the greater and parallel to these two forces.

Moreover, we have P+Q, or S : Q : : AB : AC (20).

Hence, 2d, the distances of the point of application A of this resultant from the two points C, B, are reciprocally proportional to the forces S, Q.

Remark.

25. If the ratios of the two forces S, Q, and the length of the line BC, were given in numbers, and

it were desired to find the distances of the point A from the points B, C, the preceding proportion could not be employed directly, because only the first two terms of this proportion would be known; but it is easy to deduce the following from it:

$$S-Q : Q :: AB-AC \text{ or } BC : AC,$$

in which the three first terms are known.

The distance AB can be found by another proportion,

$$S-Q : S :: AB-AC \text{ or } BC : AB,$$

which likewise is deduced from the first.

Corollary IV.

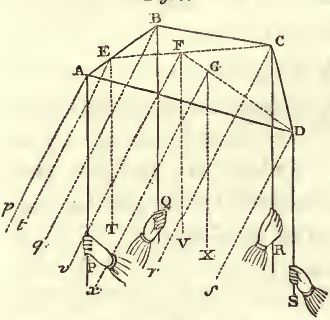
26. If the two forces S, Q, which are parallel, and act in contrary directions, be equal to each other, 1st, their resultant P, which is equal to $S-Q$ (24), becomes zero; 2d, in the proportion $S-Q : Q :: BC : AC$, the second term being infinitely great compared with the first which is zero, the fourth term is also infinitely great compared with the third. Hence the point of application A of the resultant P is at an infinite distance from the point C; hence, to produce an equilibrium between the two forces S, Q, it would be necessary to apply to the inflexible straight line a zero force, whose line of direction should be at an infinite distance; which is not absurd, but which cannot be executed.

We perceive, then, that it is impossible, by means of a single force, to produce an equilibrium between two equal and parallel forces, acting in contrary directions; but it will be demonstrated (51), that, by means of two forces, an equilibrium can be produced between them in an infinite number of ways.

PROBLEM.

27. *Any number of parallel forces* P, Q, R, S, ... *which act in the same direction, being applied to the points* A, B, C, D, ... *of given position, and connected together in an invariable manner, to determine the resultant of all these forces.*

Fig. 7.



SOLUTION.—Considering first any two of these forces, such as P and Q, determine their resultant T (18); this resultant will be equal to P+Q; its direction will be parallel to that of the forces P, Q, and we will find its point of application E, by the following proportion (22):

$$P+Q : Q :: AB : AE.$$

In place of the two forces P, Q, substitute their resultant T; then having drawn the line EC, determine the resultant V of the two forces T, R; this resultant V will be also that of the three forces P, Q, R; its intensity will be T+R or P+Q+R; and find on EC its point of application F by the proportion

$$T+R \text{ or } P+Q+R : R :: EC : EF.$$

In place of the three forces P, Q, R, let their resultant V be substituted, and having drawn the line FD, the resultant X of the two forces V, S, will be found; this resultant X will be also that of the four forces P, Q, R, S; its magnitude will be V+S, or P+Q+R+S, and

its point of application G upon FD will be found by the proportion

$$V+S, \text{ or } P+Q+R+S : S :: FD : FG.$$

By continuing in this manner, the position of the general resultant of all the forces will be found, whatever their number may be; and the intensity of this resultant will be equal to the sum of all these forces.

Corollary I.

28. Hence, by supposing the point G to be connected with the other points A, B, C, D, ... in an invariable manner, an equilibrium will be produced between all the other forces P, Q, R, S, ... by applying to the point G a force parallel to the first, which acts in the contrary direction, and which is equal to their sum

$$P+Q+R+S, \ldots.$$

Corollary II.

29. If among the parallel forces P, Q, R, S, . . . some should act in one direction, and others in the contrary direction, we could determine, first (27), the partial resultant of all which act in one direction, and then the partial resultant of all which act in the contrary direction. Thus all the forces would be reduced to two others acting in opposite directions; and by determining, by the process of No. 24, the resultant of these last two forces, supposed to be unequal, we should have the general resultant, and consequently the force which, being applied in the contrary direction, would produce an equilibrium among all the proposed forces; if these

forces should be reduced to two parallel and equal forces, we have seen (26), that it is impossible to produce an equilibrium between them.

The general resultant being equal to the difference of the two partial resultants (24), and each of these being equal to the sum of those which compose it (27), it follows that the general resultant is equal to the excess of the sum of the forces which act in one direction over the sum of those which act in the opposite direction.

Corollary III.

30. If the forces P, Q, R, S, . . . without ceasing to be parallel and without changing in intensity, had another direction and became p, q, r, s, . . . the resultant t of the first two would also pass through the point E, and would be equal to the sum $p+q$. Likewise the resultant v of the three forces p, q, r, would pass through the point F, and would be equal to the sum $p+q+r$. So also the resultant x of the four forces p, q, r, s, would pass through the point G, and would be equal to the sum $p+q+r+s$, and so on. Hence the general resultant of all the forces p, q, r, s, would pass through the same point as the resultant of the first forces P, Q, R, S, ...

Hence it appears that, when the intensities and points of application remain the same, the resultant of these forces always passes through a certain identical point, whatever may be their direction; and the intensity of this resultant is always equal to their sum.

The point through which the resultant of the parallel forces always passes, whatever may be their direction, is named the *centre of parallel forces.*

It is easy to see that if the points of application A, B, C, D, . . . of the parallel forces P, Q, R, S, . . . are in the same plane, the centre of these forces is also in this plane; for this plane contains the line AB, and consequently the point E of this line, which is the centre of the forces P, Q: it contains also the line EC, and consequently the centre F of the forces P, Q, R; it contains the line FD, and consequently the centre G of the forces P, Q, R, S, and so on.

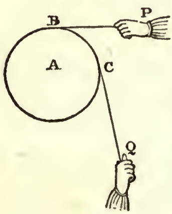
It may be demonstrated in like manner that, if the points of application are upon the same straight line, the centre of the parallel forces is also upon this line.

LEMMAS.

I.

31. *If a power* P *be applied to the circumference of a circle movable about its centre* A, *and in a direction* BP *tangent to the circumference, this force tends to turn the circle about its centre, as though it were applied at any other point* C, *and in a direction* CQ *tangent to the same circumference.*

Fig. 8.



II.

A power P, *applied along the direction of a line on which there is a fixed point, is destroyed by the resistance of this point.* (Nos. 2 *and* 3).

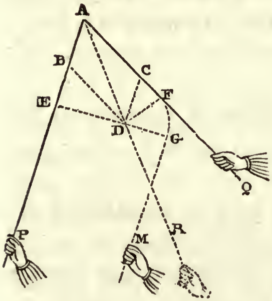
THEOREM.

32. *When the directions of two forces* P, Q, *are contained in the same plane, and intersect in the same point* A, *if we lay off in these directions the lines* AB, AC, *proportional to these forces, so that we have*

P : Q : : AB : AC,

and complete the parallelogram ABCD, *the resultant of these two forces will be in the direction of the diagonal* AD *of the parallelogram.*

Fig. 9.



DEMONSTRATION.—Suppose, for an instant, that the point D of the diagonal AD is an immovable obstacle; from this point let fall the perpendiculars DE DF upon the directions of the two forces; the triangles BED, CFD will be similar, because the angles at B and C being equal to A, will be equal to each other, and we shall have

DC : DB : : DF : DE.

Now, we have by supposition

P : Q : : AB : AC, or : : DC : DB.

Hence we shall have

P : Q : : DF : DE.

From the point D, as a centre, and with a radius DF, describe the arc FG, terminating in the prolongation of ED at G; then, regarding this arc and the line EG as inflexible lines, and connected in an invariable manner at the point A, let us conceive the force P to be applied at the point E along its direction, and a force M, equal to the force Q, to be applied to the point G, in a direction parallel to AP, and consequently tangent to the arc FG. This being granted, since M=Q and DF=DG, we will have

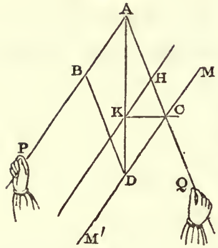
$$P : M :: DG : DE.$$

Hence (18) the resultant of the two parallel forces P, M, will pass through the fixed point D, and will be destroyed by the resistance of this point; hence these two forces will be in equilibrium around this point.

Now the force Q, whose direction is tangent to the arc FG, and which we may regard as applied to the point F on its direction, tends to turn this arc in the same manner as the force M (31), and may be substituted for this latter force to counter-balance the force P: hence the two forces P, Q, will also be in equilibrium around the fixed point D; hence their resultant will be destroyed by the resistance of this point, and consequently, (31) the direction of this resultant will pass through the point D.

But the resultant of the two forces P, Q, should pass through the point of intersection A of their directions (4); hence this resultant will be in the direction of the diagonal AD.

Fig. 10.



33. Another Demonstration.* Suppose that the force Q acts at the point C on its direction, being invariably fixed at the point A, and that there is applied to the same point C, and in the opposite directions CM, CM′, two forces M, M′, each equal to Q, the effect of the two forces P and Q will be the same as that of the four forces P, Q, M, M′, since the last two, M and M′, destroy each other, being equal and opposed. Now, these four forces form two couples; the one, Q and M intersect at the point C; the other is composed of two parallel and unequal forces P, M′, applied to the line AC; the resultant of the two equal forces Q, M, is divided along the line CK, which divides (12) the angle MCQ into two equal parts; the resultant HK of the two forces P, M′, applied to the points A, C, of the line AC, passes through a point H of this line, so that we have (18)

$$P : M', \text{ or } Q :: HC : HA;$$

moreover, it is parallel to the direction AB of the force P. Hence the point K, the intersection of the two resultants CK, HK, is a point of the resultant of the four forces P, Q, M, M′, and consequently of the first two P and Q. The point K is upon the direction of the diagonal AD of the parallelogram constructed upon AB and AC, as sides; thus, by construction, the angle HCK is equal to the angle KCD. Now, the angles KCD and CKH

* This demonstration differs but little from that given by M. Poinsot, in the first edition of his *Statique*, in the year 1803.

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are equal, being alternate internal angles; hence, in the triangle CHK, the angles K and C are equal; hence it follows, that the line KH has the same length as the line HC; but we have the proportion:

P : Q : : HC : HA;

hence we will have

P : Q : : KH : HA,

and since

P : Q : : CD : AC,

the ratio of the lines AC and CD is the same as that of the lines AH and HK; hence the three points A, K, D, are in a straight line, and this line is the diagonal of the parallelogram constructed upon AB and AC as sides.

Corollary I.

34. If from any point D (Fig. 9), taken upon the direction AD of the resultant of the two forces P, Q, the lines DB, DC be drawn parallel to the directions of these forces, a parallelogram ABCD will be formed, whose sides AB, AC, will be proportional to the forces P, Q, that is to say, we will have:

P : Q : : AB : AC, or : : DC : DB.

For if these sides were not proportional to the forces, their resultant would be in the direction of the diagonal of the parallelogram whose sides would be proportional to these forces (32), and not in the direction AD, which would be contrary to the supposition.

Corollary II.

35. If from any point D (Fig 9), taken upon the direction AD of the resultant of the two forces P, Q,

the perpendiculars, DE, DF, be drawn upon the directions of these two forces, these perpendiculars will be to each other reciprocally as the forces P, Q.

For we have just seen (34) that

$$P : Q :: DC : DB;$$

and the similar triangles DBE, DCF give

$$DC : DB :: DF : DE;$$

hence

$$P : Q :: DF : DE.$$

THEOREM.

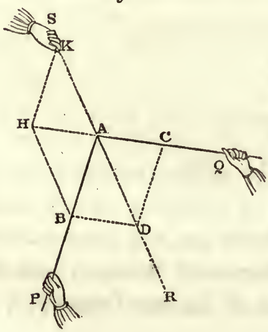
36. *When the directions of the two forces* P, Q, *are contained in the same plane, and coincide in a point* A, *if the lines* AB, AC, *be laid off on these directions proportional to these forces, so that*

$$P : Q :: AB : AC,$$

and the parallelogram ABDC *be completed, the resultant* R *of these two forces will be represented in intensity and direction by the diagonal* AD *of the parallelogram; that is to say, we shall have:*

$$P : Q : R :: AB : AC : AD.$$

Fig. 11.



DEMONSTRATION. We have already seen (32) that the resultant of the two forces P, Q, will be in the direction of the diagonal AD of the parallelogram; it is only necessary to show that its intensity will be represented by this diagonal.

Let there be applied to the point A, a force S, equal and directly opposed to the resultant R; this force will be in the direction of the prolongation of the diagonal DA, and the three forces P, Q, S, will be in equilibrium. Hence the force Q will also be equal and directly opposed to the resultant of the two other forces P, S; and consequently this last resultant will be in the direction of the prolongation of the line CA. Let CA be laid off on its prolongation from A to H; draw the line HB, which will be parallel to AD, and consequently to the direction of the force S; and through the point H draw HK parallel to the direction of the force P: the two forces P, S, will be to each other as the sides AB, AK, of the parallelogram ABHK (34), that is to say,

$$P : S :: AB : AK \text{ or } HB.$$

Now, by reason of the parallelogram, $AD=BH$; moreover, the two forces S and R are equal; hence we have,

$$P : R :: AB : AD.$$

But by supposition

$$P : Q :: AB : AC.$$

Hence, by combining these last proportions, we will have

$$P : Q : R :: AB : AC : AD.$$

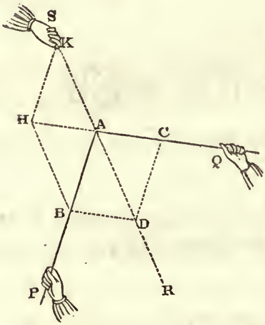
Corollary I.

37. If the two forces P, Q, be applied to the point A, equilibrium will be produced by applying to the same point a third force in the direction AK, and proportional to the diagonal AD; for this force will be equal and directly opposed to the resultant of the two forces P, Q.

If the forces P, Q be applied to other points of their direction, equilibrium can be produced by applying to any point of the line AD, and in the direction DA, a force proportional to DA, provided the point of application of this last force is connected in an invariable manner with the points of application of the two forces P, Q.

COROLLARY II.

Fig. 11.



38. A force R, of given intensity and direction, may be always decomposed into two other forces P, Q, in the direction of the given lines AP, AQ, provided these directions and that of the force R are included in the same plane and coincide in the same point A.

For this purpose, we will represent the force R by a part AD of its direction; then drawing through the point D the lines DC, DB, parallel to the given directions AP, AQ, we will form a parallelogram ABCD, whose sides AB, AC, will represent the required forces P, Q; for (36) the resultant of these two forces will have the same intensity and the same direction as the force R.

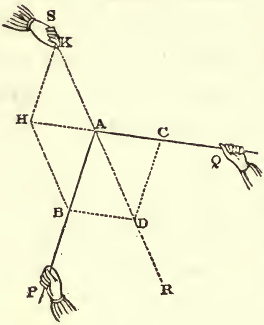
The intensities of the two forces P, Q, may be found by means of the proportions:

$$AD : AB :: R : P.$$

$$AD : AC :: R : Q.$$

Corollary III.

Fig. 11.



39. In the triangle ABD (Fig. 11), the sides AB, BD, AD, are proportional to the sines of the angles CAD, DAB, BAC, formed by the directions of the forces P, Q, R; hence it follows, that we shall have the proportion

$$P : Q : R :: \sin(\widehat{Q.R}) : \sin(\widehat{P.R}) : \sin(\widehat{P.Q})$$

the notation $(\widehat{Q.R})$, &c., being the angle of the two forces Q and R, &c.

The same triangle ABD gives

$$\overline{AD}^2 = \overline{AB}^2 + \overline{BD}^2 \pm 2AB \times BD \cos ABD;$$

hence

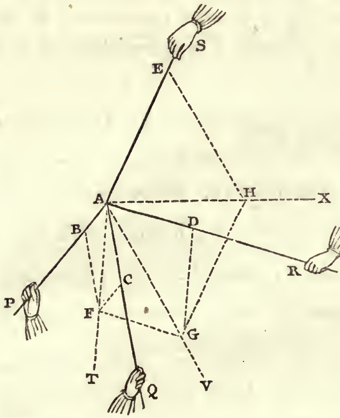
$$R^2 = P^2 + Q^2 \pm 2PQ \cos(\widehat{P.Q}).$$

PROBLEM.

40. *To determine the resultant of any number of forces* P, Q, R, S . . . . *whose directions, contained or not contained in the same plane, meet in the same point* A.

Solution. Lay off upon the directions of all the forces, from the point A, the lines AB, AC, AD, AE . . . proportional to their intensities; then, considering first any two of these forces, as P, Q, complete the parallelogram ABFC, whose diagonal AF will represent in intensity and direction the partial resultant T of these two forces (36).

Fig. 12.



Instead of the forces P, Q, take their resultant T, and considering the two forces T, R, complete the parallelogram AFGD, whose diagonal AG will represent in intensity and direction the resultant V of the two forces T, R, which will be that of the three forces P, Q, R.

In like manner, instead of the forces P, Q, R, take their resultant V, and considering the two forces V, S, complete the parallelogram AGHE, whose diagonal AH will represent, in intensity and direction, the resultant X of the forces V, S, which will also be that of the four forces P, Q, R, S.

By continuing in this way, the direction and intensity of the general resultant of all the forces P, Q, R, S . . . will be found, whatever may be their number.

Corollary.

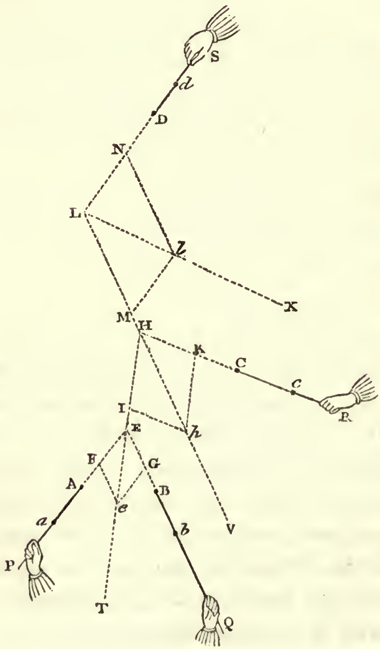
41. If all the forces P, Q, R, S . . . . be applied to the point of coincidence A of their directions, in order to produce an equilibrium, find first the intensity and direction of their resultant (40); then apply to the point A a force equal and directly opposed to it. But if the forces be applied to other points of their directions, connected together in an invariable manner, an equilibrium will be produced by applying to any invariable point in the direction of their resultant, a force equal

and directly opposed to this resultant, provided the point of application of this force is also connected in an invariable manner with the points of application of the forces P, Q, R, S . . . .

PROBLEM.

42. *To determine the resultant of any number of forces, whose directions included in the same plane do not meet in the same point, whose points of application* A, B, C, D . . . . *are connected together in an invariable manner, and whose intensities are represented by the parts* Aa, Bb, Cc, Dd, ... *of their directions.*

Fig. 13.



SOLUTION. Having prolonged the directions of any two of the forces, for instance P, Q, until they meet somewhere in a point E, lay off from E to F and from E to G the lines Aa, Bb, which will represent these forces; and complete the parallelogram EFeG, whose diagonal Ee will represent in intensity and direction the resultant T of the two forces P, Q (36).

Instead of the forces P, Q, take their resultant T, and prolong its direction as well as that of the force R until they meet somewhere in a point H; lay off the line E*e* from H to I, the line C*c* from H to K, and complete the parallelogram HI*h*K, whose diagonal H*h* will represent in intensity and direction the resultant V of the two forces T, R, which will be also that of the three forces P, Q, R.

In like manner, instead of the three forces P, Q, R, take their resultant V, and prolong its direction, as well as that of the force S, until they meet in a point L; then laying off from L to M and from L to N the lines H*h*, D*d*, which represent the forces V and S, complete the parallelogram LM*l*N, whose diagonal L*l* will represent the resultant X of these two forces, which will also be that of the four forces P, Q, R, S.

By thus continuing, the intensity and direction of the general resultant of all the proposed forces will be found, whatever may be their number.

COROLLARY.

43. Hence, when several forces, directed in the same plane, are applied to points connected together in an invariable manner, these forces always have one resultant: thus it is possible to produce an equilibrium by means of a single force, except in the case where the direction of one of these forces being parallel to that of the resultant of all the others, the force and the resultant would always be equal to each other and would act in contrary directions; for we have seen (26) that, in order to produce an equilibrium in that case, it would be necessary to apply a zero force in a line situated at an infinite distance; which is impracticable.

THEOREM.

44. *If three forces* P, Q, R, *be represented in intensity and direction by the three sides* AB, AC, AD, *adjacent to the same angle of a parallelopipedon* ABFEGD, *so that*

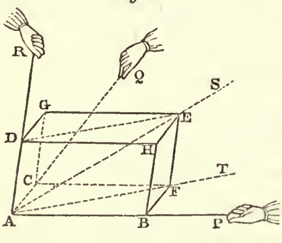
$$P : Q : R :: AB : AC : AD,$$

their resultant S *will be represented in intensity and direction by the diagonal* AE *of the parallelopipedon adjacent to the same angle, and we shall have*

$$P : Q : R : S :: AB : AC : AD : AE.$$

DEMONSTRATION. In the plane ABCF, which contains the directions of the two forces P, Q, draw the diagonal AF; also draw the diagonal DE in the opposite face DHEG: these two diagonals will be equal and parallel; for the two sides AD, EF of the parallelopipedon at the extremities of which they terminate are parallel and equal: hence AFED will be a parallelogram. This done, the two forces P, Q, being represented in intensity and direction by the sides AB, AC, of the plane ABFC, which is a parallelogram, their resultant T will be represented in magnitude and direction by the diagonal AF (36), and we shall have

Fig. 14.



$$P : Q : T :: AB : AC : AF.$$

Likewise the two forces T, R, being represented by the sides AF, AD, of the parallelogram AFED, their resultant

S, which will be also that of the three forces P, Q, R, will be represented by the diagonal AE of the same parallelogram, and we shall have

T : R : S : : AF : AD : AE.

Hence, combining the two proportions, we will obtain

P : Q : R : S : : AB : AC : AD : AE.

Now the diagonal AE is also the diagonal of the parallelopipedon; hence the resultant of the three forces P, Q, R will be represented in intensity and direction by the diagonal of the parallelopipedon.

Corollary I.

45. A force S of given intensity and direction can always be decomposed into three other forces P, Q, R, in the direction of the given lines AP, AQ, AR, not included in the same plane, provided these three directions and that of the force S coincide in the same point A.

Thus, through the three directions, considered two and two, draw the three planes BAC, CAD, DAB; represent the force S by a part AE of its direction; and through the point E draw three other planes EGDH, EHBF, EFCG, respectively parallel to the first three. These six planes will be the faces of a parallelopipedon, of which AE will be the diagonal, and of which the sides AB, AC, AD, taken upon the three given directions, will represent the intensities of the three required forces P, Q, R; for (44) the resultant of these three forces will have the same intensity and direction as the force S.

Or else, draw through the point E three lines parallel to the directions AP, AQ, AR; and the parts EF, EH, EG, of these sides, included between the point E and the planes BAC, CAD, DAB, will represent the intensities of

the required forces P, Q, R; for these lines, being three sides of the parallelopipedon, are respectively equal to the other sides AB, AC, AD, which are parallel to them.

Corollary II.

46. When the three forces P, Q, R, are perpendicular to each other, the resultant S is the diagonal of a rectangular parallelopipedon, whose three sides adjacent to the same summit of an angle are equal to the three forces P, Q, R; the intensity of this resultant is, in this case, expressed by

$$\sqrt{P^2+Q^2+R^2}.$$

Corollary III.

47. Whatever may be the number of forces P, Q, R, S, . . . applied to the fixed points A, B, C, D, . . . we can always conceive the system of three right lines, perpendicular to each other, to be transferred to the points of application of the forces so as to occupy positions parallel to their former positions, and each of these forces to be decomposed into three others, in the directions of the three rectangular lines passing through the point of application; then all the forces P, Q, R, S, . . . will be decomposed into three systems of forces, so that all the forces of the same system will be reduced to a single force along the same direction; hence all the forces P, Q, R, S, will have three resultants parallel to the three rectangular lines, fixed and determined in position with reference to these forces. (See No. 53).

Corollary IV.

48. Call S, S′, S″ . . . . the forces which act upon a determined point; and drawing through this point three lines fixed and perpendicular to each other, each of the forces S, S′, S″, . . . will be decomposed into three others p, q, r, in the direction of the rectangular lines.

In like manner calling p', q', r', the three component forces of the force S′; and p'', q'', r'', the three component forces of the force S″, &c.; the resultant of all the forces S, S′, S″ will be the diagonal of a rectangular parallelopipedon, whose three sides adjacent to the same angle will be,

For the first, $p+p'+p''+\ldots.$;

For the second, $q+q'+q''+\ldots.$;

For the third, $r+r'+r''+\ldots.$.

Hence the expression for this resultant will be

$$\sqrt{(p+p'+p''+\ldots)^2+(q+q'+q''+\ldots)^2+(r+r'+r''+\ldots.)^2}.$$

THEOREM.

49. *Two forces in the direction of lines which do not meet cannot be reduced to a single force equivalent to them.*

Demonstration. Let P and Q be the two forces whose directions do not meet. If a third force R causes an equilibrium to subsist between them, any two fixed points, one being taken upon the direction of this force R, and the other upon the direction of the force P, will necessarily destroy the force Q: now these two points

cannot be so taken that the line which unites them will not meet the force Q; hence this force will not be destroyed; hence it is absurd to suppose that the two forces P and Q can have a single resultant R.

THEOREM.

50. *All the forces* P, Q, R, S, . . . *applied to the points* A, B, C, D, . . . *joined together in an invariable manner, may in general be reduced to two forces in the directions of lines which do not meet.*

DEMONSTRATION. Having extended the lines in the directions of the forces P, Q, R, . . . until they cut a plane having a fixed and determined position with reference to these lines, we may consider the points of intersection as the points of application of the forces; now each force may be decomposed into two, one situated in the plane and the other perpendicular to this plane: all the forces directed in the plane will have one resultant; the forces perpendicular to the plane, and consequently parallel to each other, will have another resultant. In some particular cases, these two resultants meet, and all the proposed forces P, Q, R, S, . . . will be reduced to a single one; but in general they will not meet: hence we shall have two forces, one situated in a plane assumed arbitrarily, and the other perpendicular to this plane, which will produce an equilibrium with any number of forces P, Q, R, S, ... applied to the points A, B, C, D . . . . It is necessary to except from this general conclusion the particular case which we are

about to examine, and which takes place when the forces situated in the plane and the forces perpendicular to the plane, are reduced to one or more couples of equal and parallel forces applied in opposite directions to the same right line.

Corollary.

51. When two forces act along the direction of lines which do not meet, there are an infinite number of systems of two forces acting in the direction of other lines which do not meet, whose action is equivalent to that of the first two forces: in fact, any force may be decomposed into two other forces, one perpendicular to a plane assumed at will, and the other situated in this plane; hence any two forces are equivalent to two other forces, one being situated in the plane assumed arbitrarily, and the other perpendicular to this plane.

PROBLEM.

52. *Two forces being given in the direction of lines which do not meet, to find two others equivalent to them, one of which is in the direction of a line having a given position.*

Solution. Let P and Q be the two given forces; having drawn a plane perpendicular to the line of given position, decompose the forces P and Q into two others P′ and Q′, one having its direction in this plane and the other perpendicular to the same plane; decompose the force Q′, parallel to the given line, into two others q', q'', parallel to Q′, of which the one q' will pass through the

given line, and the other q'' through a point of the force P′; the two forces P′, q'', meeting in the same point, will be reduced to a single force q; hence the two forces P and Q will be transformed into two other equivalent forces q, q', the latter of which, q', will pass through a given line.

53. *Examination of a particular case of the composition of forces applied to given points* A, B, C, D, . . . *invariably joined together.*

Propositions.

Having decomposed each of the forces P, Q, R, S, . . . applied to the points A, B, C, D . . . into two others, one situated in a given plane K, and the other perpendicular to this plane, let S and T be the resultants of these two systems of forces; it may happen that the first system of forces, instead of having a single force S for resultant, will only reduce to a couple of forces $+s, -s$, equal, parallel, opposite, and applied to the same line; in this case, the three forces, T, $+s, -s$, will be equivalent to two forces beyond the plane of the couple $+s$, $-s$: likewise, if instead of a single resultant T, the forces perpendicular to the plane K should reduce to a single couple $+t$, $-t$, (designating by this expression two equal and parallel forces, opposed and applied to the same line), the three forces S, $+t, -t$, will also be equivalent to two forces, as in the preceding case: finally, if all the forces acting in the plane K and perpendicularly to this plane, should be reduced to two couples $+s, -s$, and $+t, -t$, these two couples would compose a single one.

The three propositions which have just been announced are included in the two following:

1st. A force T and a couple $+s, -s$ are equivalent to two forces in the directions of lines which do not meet;

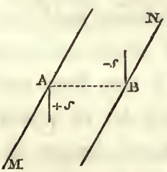
2d. Two couples $+s, -s$ and $+t, -t$ are equivalent to a single couple of a like nature: such for example as $+r, -r$.

Demonstration of these two Propositions.

54. 1st Proposition. A force T and a couple $+s, -s$ will combine into two forces: thus, the plane of the couple being prolonged intersects the force T in a point which may be regarded as the point of application of the force T; drawing any line through this point and in the plane of the couple, and regarding this line as fixed with reference to the three forces T, $+s, -s$ which are applied to it, decompose the force T into two other parallel ones t, t', which will have upon the fixed line the same points of application as the forces $+s, -s$. The forces t, s, meeting in the same point, will have a resultant; so also will the forces $t', -s$: they will have a second resultant; these two resultants evidently will be equivalent to the three forces T, $+s, -s$.

55. If the plane of the couple were parallel to the force T, it would be necessary to decompose the forces of the couple $+s$ and $-s$ parallel to T. Suppose this couple is applied to the line AB (Fig. 13 a) perpendicular to this line; drawing through their points of application A and B, the lines AM, BN, parallel to the direction of the force T, decompose the force $+s$ into two

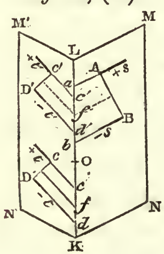
Fig. 13. (a.)



others in the directions AM and AB, the force $-s$ into two others BN and BA; the forces along the direction AB will be destroyed; the force along the direction AM or BN, and the force T which is parallel to it, will combine to form a single force: hence, in this case, the three forces T, $+s$, $-s$, will reduce to two, one in the plane of the couple $+s$, $-s$, and the other parallel to this plane.

56. 2D PROPOSITION. Two couples $+s$, $-s$ and $+t$, $-t$, situated in any planes whatever, will combine into a single couple $+r$, $-r$: thus, the planes of these couples being prolonged will meet in the direction of a line which may be considered as invariably joined to the points of application of the forces which compose the two couples. Let KL (*Fig.* 13 *b*) be this line, the intersection of the two planes LKMN, LKM′N′, one of which contains the couple $+s$, $-s$, applied to the line AB, the other the couple $+t$, $-t$, applied to the line CD; the directions of the forces $+s$, $-s$, $+t$, $-t$ intersect this line in the points a, b, c, d; dividing the lines ab, cd into two equal parts, and marking the middle points f', f, we may transfer the couple $+t$, $-t$, applied to the line CD, parallel to itself, so that the points f, f' will coincide; we will then have a new couple $+t'$, $-t'$. It is necessary, in the first place, to demonstrate that this second couple, composed of forces equal and parallel to those of the first and applied to the line C′D′ equal to CD, will be in equilibrium with it, and that in general we cannot change the condition of equilibrium of two couples, by transferring

Fig. 13, (*b.*)



one of the couples parallel to itself in its plane. Now, this proposition is evident, for the point O being the middle of the line $c'd$, the two forces $+t', -t$, as well as the two forces $-t', +t$, which act at equal distances from this point O, are in equilibrium; hence the second couple may be substituted for the first. Now, the force $+t'$ is decomposable into two other parallel forces passing through the points a and b; the force $-t'$ also may be decomposed into two others passing through the points b and a; and since $c'a=d'b$, the component forces of these two forces $+t'$ and $-t'$, will be equal, and will differ only in direction: hence, each of the three forces meeting in the point a will be equal to one of the three forces meeting in the point b; hence, the resultants of the two systems of forces applied to the points a and b will be equal and opposed; hence it follows that the two couples $+s, -s$ and $+t, -t$ will be reduced to a single one $+r, -r$.

If the forces $+t, -t$ were parallel to the line LK, the couple $+s$, $-s$ (55) might be changed into another $+s', -s'$, whose forces would be directed parallel to the line LK, and the four parallel forces $+t, +s', -t, -s'$ would be reduced to two equal and opposed forces $+(t+s')$ and $-(t+s')$.*

* See the theory of couples in the *Statique* of M. Poinsot, 6th edition, 1834.

CHAPTER SECOND.

OF MOMENTS.

57. Two kinds of *moments* are to be considered. The *moment* of a force referred to a *point* is the product of this force multiplied by the perpendicular let fall from the point upon the force; the *moment* of a force referred to a *plane*, is the product of this force multiplied by the distance of its point of application from the plane: this second kind of *moment* does not change, even though the forces vary in direction; they differ in this condition from the *moments* of the first kind, which are independent of the points of application of the forces whose direction is constant.

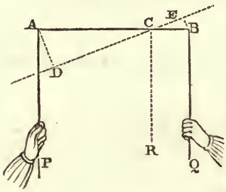
58. When the moments of several forces, referred to the same point, are considered, this point is named the *centre of moments.*

59. Hence it follows, if the intensity of a force be known, and its moment referred to a centre or a plane, and if the plane be parallel to the force, we will obtain the distance of the centre, or the plane, from the direction of the force, by taking the quotient of the moment divided by the force; if the moment and the distance be known, we shall obtain the intensity of the force by taking the quotient of the moment divided by the distance.

COROLLARY.

60. When the two parallel forces P, Q, act in the same direction, their moments, referred to any point C in the direction of their resultant, are equal.

Fig. 15.



For if through the point C the line AB be drawn perpendicular to the direction of the two forces, and terminate in A and B, this line will be divided, at the point C, into two parts reciprocally proportional to the forces P, Q (18): that is to say, we shall have

$$P : Q :: BC : AC.$$

Hence, the product of the extremes being equal to the product of the means, we shall have

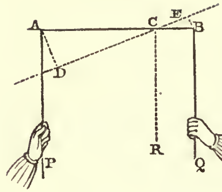
$$P \times AC = Q \times BC.$$

THEOREM.

61. *If at the extremities* A, B *of an inflexible right line, two parallel forces* P, Q *be applied which act in the same direction, and if through the point of application* C *of their resultant, a right line* DE *be drawn in any plane, upon which the perpendiculars* AD, BE *be dropped from* A *and* B, *we shall have*

$$P \times AD = Q \times BE.$$

Fig. 15.



DEMONSTRATION. The right angled triangles ADC, BEC, which are similar, because the angles opposite the summit C are equal, give

$$BC : AC :: BE : AD.$$

Now we have (18)

$$P : Q :: BC : AC.$$

Hence

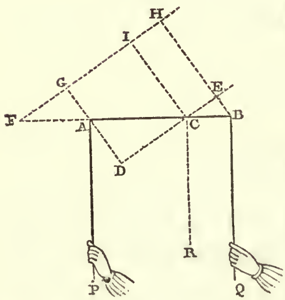
$$P : Q :: BE : AD;$$

and, the product of the extremes equalling that of the means,

$$P \times AD = Q \times BE.$$

THEOREM.

Fig. 16.

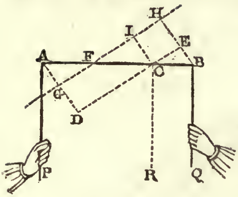


62. *Two parallel forces* P, Q, *which act in the same direction, being applied to the points* A, B *of an inflexible right line, and through a point* F *of this line, the line* FH *being drawn in any plane:*

1*st. If the point* F *be taken upon the prolongation of* AB, *and if from the points* A, B, *and from the point of application* C *of the resultant, the perpendiculars* AG, BH, CI, *be dropped upon* FH, *we shall have*

$$R \times CI = Q \times BH + P \times AG;$$

Fig. 17.

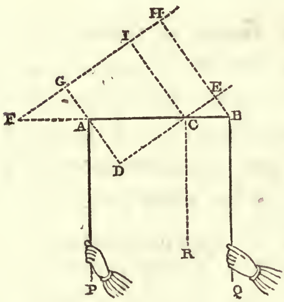


2d, If the point F *be taken between* A *and* B, *we shall have*

$$R\times CI=Q\times BH-P\times AG.$$

DEMONSTRATION. Through the point C, draw DE parallel to FH, intersecting the perpendiculars AD, BH in the points D, E; we have DG=CI=EH, besides (61), P×AD=Q×BE.

Fig. 16.



Now, in the first case, the resultant R being equal to the sum of the two forces P, Q (18), we shall have

$$R\times CI=(Q+P)\times CI,$$

or

$$=Q\times HE+P\times GD.$$

But we have GD=AD+AG; hence

$$R\times CI=Q\times HE+P\times AD+P\times AG;$$

or placing the product Q×BE, in place of P×AD which is equal to it, we shall have:

$$R\times CI=Q\times HE+Q\times BE+P\times AG;$$

or finally

$$R\times CI=Q\times BH+P\times AG.$$

In the second case we have likewise

$$R\times CI=Q\times HE+P\times GD.$$

But we have GD=AD−AG; hence

$$R\times CI=Q\times HE+P\times AD-P\times AG;$$

or substituting for P×AD its value Q×BE, we shall have

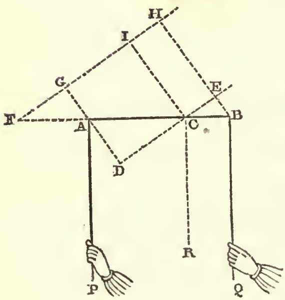
$$R \times CI = Q \times HE + Q \times BE - P \times AG;$$

and finally,

$$R \times CI = Q \times BH - P \times AG.$$

Corollary I.

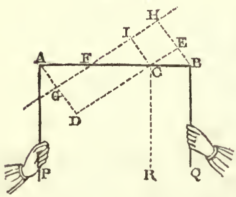
Fig. 16.



63. Hence it follows, 1st. When the two points A, B are on the same side of the line FH, the distance CI of the point C of the resultant from this line will be, since $R = P + Q$,

$$CI = \frac{Q \times BH + P \times AG}{P+Q}.$$

Fig. 17.



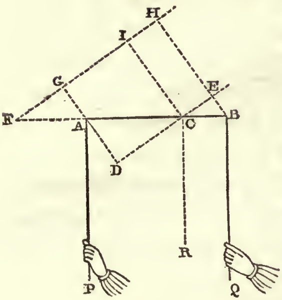
2d. When the points A, B are placed on different sides of the line FH, this distance will be

$$CI = \frac{Q \times BH - P \times AG}{P+Q}.$$

In this case the point C will be placed, with reference to the line GH, on the same side as the force Q, for which the product Q×BH is the greater.

Corollary II.

Fig. 16.



64. If the line FH is perpendicular to AB, the lines AG, BH, CI will all three be in the direction of AB, and the proposition enunciated in the preceding theorem will still take place. In this case we shall have

$$AG=AF,\ BH=BF,\ CI=CF.$$

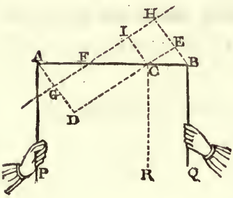
Then we shall have for *Fig.* 16,

$$R\times CF=Q\times BF+P\times AF;$$

and for *Fig.* 17,

$$R\times CF=Q\times BF-P\times AF.$$

Fig. 17.



As to the distance CF from the point F to the point of application C of the resultant $R=P+Q$, it will be for *Fig.* 16,

$$CF=\frac{Q\times BF+P\times AF}{P+Q}.$$

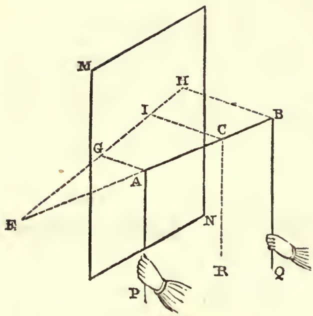
and for *Fig.* 17,

$$CF=\frac{Q\times BF-P\times AF}{P+Q}.$$

THEOREM.

65. *Two parallel forces* P, Q (*Figs.* 18 *and* 19), *which act in the same direction, being applied to the points* A, B *of an inflexible right line, and having drawn a plane* MN, *through a point* F *of this line, parallel to their directions:*

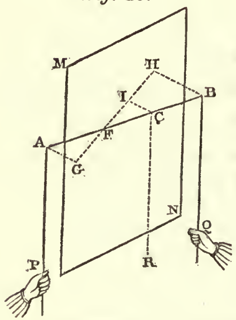
Fig. 18.



1st. *If the point* F *be taken upon the prolongation of* AB, *the sum of the moments of the two forces* P, Q, *referred to the plane* MN, *will be equal to the moment of their resultant* R*: that is to say, if from the points* A, B, *and from the point of application* C *of the resultant, the perpendiculars* AG, BH, CI, *be let fall upon the plane, we shall have*

$$R\times CI=Q\times BH+P\times AG.$$

Fig. 19.



2d. *If the point* F *be taken between* A *and* B, *the difference of the moments of the forces* P, Q *will be equal to the moment of their resultant: that is to say, we shall have*

$$R\times CI=Q\times BH-P\times AG.$$

DEMONSTRATION. The three lines AG, BH, CI, perpendicular to the same plane MN, are parallel to each other; moreover they pass through the three points A, B, C, of the same line: hence they are in the same plane drawn through A, B; hence their feet G, H, I, and the point F are in the same plane. But the four points F, G, H, I, are also in the plane MN; hence they are in the intersection of two different planes, and consequently in a straight line. Now draw the line FGIH:

it will intersect the lines AG, BH, CI, at right angles; for it will be in the plane MN to which these lines are perpendicular, and it will pass through their feet. Hence, by considering FGIH as the line FH (*Figs* 16 and 17):

1st. When the point F (*Fig.* 18) is upon the prolongation of AB, we shall have (62)

$$R \times CI = Q \times BH + P \times AG.$$

2d. When the point F (*Fig.* 19) is between A and B, we shall have

$$R \times CI = Q + BH - P \times AG.$$

COROLLARY.

66. Hence, 1st. When the two forces P, Q, (*Fig.* 18) are on the same side of the plane MN, the distance CI of the plane from the resultant will be equal to the sum of the moments of the forces referred to the plane, divided by the resultant R, or, what is the same (18), divided by the sum P+Q of the forces: that is to say, we shall have

$$CI = \frac{Q \times BH + P \times AG}{P + Q}.$$

2d. When the plane passes between the directions of the two forces (*Fig.* 19), this distance will be equal to the difference of the moments divided by the sum of the forces: that is to say, we shall have

$$CI = \frac{Q \times BH - P \times AG}{P + Q}.$$

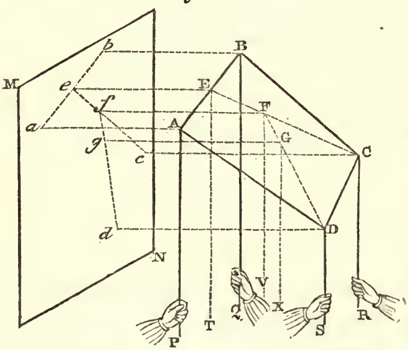
In the latter case, the resultant will be situated on the same side of the plane MN as the force whose moment is the greater.

THEOREM.

67. *If, to any number of points* A, B, C, D, . . . *situated or not in the same plane, but connected together in an invariable manner, the parallel forces* P, Q, R, S, . . . *be applied, which act in the same direction, and which are all placed on the same side of any plane* MN *parallel to their directions, the sum of the moments of all the forces referred to the plane* MN *will be equal to the moment of their resultant.*

DEMONSTRATION. Draw the line AB, and let E be the point on this line through which the resultant T of the two forces P, Q passes; draw the line EC, and let F be the point on this line through which the resultant V of the two forces T, R, passes, which will also be the resultant of the three forces P, Q, R; draw FD, and let G be the point on this line through which the resultant X of the two forces V, S, passes, which will also be the resultant of the four forces P, Q, R, S; and so on. Finally, from the points A, B, C, D . . . . and the points E, F, G, . . . let fall upon the plane MN the perpendiculars Aa, Bb, Cc, Dd, ... , Ee, Ff, Gg, ...

Fig. 20.



Then, the moment of the resultant T will be equal to the sum of the moments of its two components P, Q (65), and we shall have

$$T \times Ee = P \times Aa + Q \times Bb.$$

In like manner, the moment of the force V will be equal to the sum of the moments of its two components T, R, and we shall have

$$V \times Ff = T \times Ee + R \times Cc.$$

Hence, substituting for $T \times Ee$ its value, we get

$$V \times Ff = P \times Aa + Q \times Bb + R \times Cc.$$

Likewise the moment of the force X will be equal to the sum of the moments of its two components V, S, which will give

$$X \times Gg = V \times Ff + S \times Dd.$$

Hence, substituting for $V \times Ff$ its value, we shall have

$$X \times Gg = P \times Aa + Q \times Bb + R \times Cc + S \times Dd.$$

And so on, whatever may be the number of forces. Hence the moment of any resultant is equal to the sum of the moments of all the components; hence, &c.

Corollary I.

68. We have seen (27) that the intensity of the resultant X of the forces P, Q, R, S, ... is equal to the sum $P+Q+R+S$ . . . . of these forces: hence the distance Gg of the direction of this resultant from the plane MN is equal to the sum of the moments of all the

forces P, Q, R, S, . . . divided by the sum of all these forces: that is to say, we shall have

$$\text{G}g = \frac{\text{P}\times\text{A}a+\text{Q}\times\text{B}b+\text{R}\times\text{C}c+\text{S}\times\text{D}d}{\text{P}+\text{Q}+\text{R}+\text{S}}.$$

Corollary II.

69. Hence, if an indefinite plane be drawn on the side on which the forces are placed, parallel to MN and at a distance Gg from it: that is to say, equal to

$$\frac{\text{P}\times\text{A}a+\text{Q}\times\text{B}b+\text{R}\times\text{C}c+\text{S}\times\text{D}d}{\text{P}+\text{Q}+\text{R}+\text{S}},$$

this plane will contain the direction of the resultant of all the forces P, Q, R, S . . . .; for this plane will contain all the points which, on this side, are distant from the plane MN by the quantity Gg, consequently all the points of the direction of the resultant.

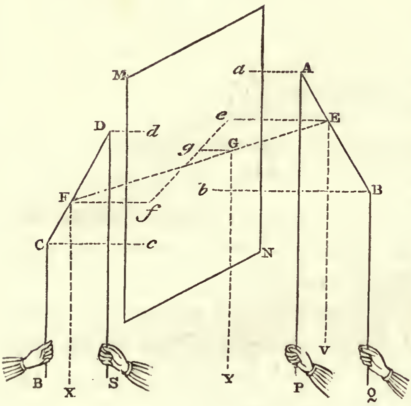
Corollary III.

70. *If the forces* P, Q, R, S, . . . *be situated on each side of the plane* MN, *the moment of their resultant, referred to this plane, will be equal to the excess of the sum of the moments of the forces which are situated on one side of the plane, over the sum of the moments of the forces which are situated on the other side.*

Thus, let V be the partial resultant of all the forces P, Q . . . which are situated on one side of the plane, whatever their number may be, and E the point of ap-

plication of this force. In like manner, let X be the partial resultant of all the forces R, S, . . . which are situated on the other side, and F the point of application of this force. By letting fall upon the plane the perpendiculars A*a*, B*b*, E*e* . . . C*c*, D*d*, F*f*, . . we have just seen (67) that we shall have

Fig. 21.



$$V \times Ee = P \times Aa + Q \times Bb \ldots,$$

and

$$X \times Ff = R \times Cc + S \times Dd \ldots .$$

Now let Y be the resultant of the two forces V, X, and G its point of application; this force will be the general resultant of all the forces P, Q, R, S, . .

This done, the two forces V, X, being situated on each side of the plane MN, the moment of their resultant is equal to the difference of their moments (65); hence, letting fall the perpendicular G*g* upon the plane, we shall have

$$Y \times Gg = V \times Ee - X \times Ff.$$

Hence, by substituting the values of these last two moments, we shall have

$$Y \times Gg = P \times Aa + Q \times Bb \ldots - (R \times Cc + S \times Dd \ldots).$$

Hence, &c.

Corollary IV.

71. Hence, in general, in whatever manner several parallel forces P, Q, R, S, . . . acting in the same direction, may be situated with reference to a plane MN, parallel to their directions, the distance Gg of their resultant from this plane is equal to the excess of the sum of the moments of the forces situated on one side of the plane, over the sum of the moments of the forces situated on the other side, divided by the sum of all the forces: that is to say, we shall have

$$Gg = \frac{P \times Aa + Q \times Bb \ldots . - (R \times Cc + S \times Dd \ldots)}{P + Q + R + S};$$

and this resultant is placed, with reference to the plane MN, on the side on which the sum of the moments is the greater.

Corollary V.

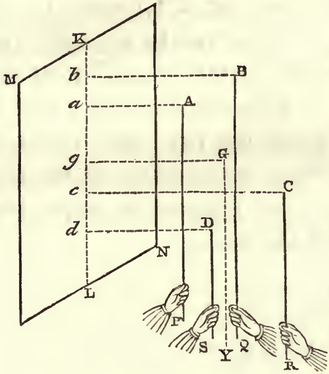
72. Hence, if on the side of the plane MN, on which the sum of the moments is the greater, we draw an indefinite plane parallel to it, which is distant by the quantity Gg, or

$$\frac{P \times Aa + Q \times Bb \ldots - (R \times Cc + S \times Dd \ldots)}{P + Q + R + S \ldots},$$

this plane will contain the direction of the resultant of all the forces P, Q, R, S, . . .

COROLLARY VI.

Fig. 22.



73. If the directions of the forces P, Q, R, S, . . . be all situated in the same plane, perpendicular to the plane MN, the lines Aa, Bb, Cc, Dd . . . . Gg will fall upon the line KL, the intersection of the two planes; and we shall then also have

$$\text{Y}\times\text{G}g=\text{P}\times\text{A}a+\text{Q}\times\text{B}b\ldots+(\text{R}\times\text{C}c+\text{S}\times\text{D}d\ldots),$$

$$\text{Y}\times\text{G}g=\text{P}\times\text{A}a+\text{Q}\times\text{B}b\ldots-(\text{R}\times\text{C}c+\text{S}\times\text{D}d\ldots),$$

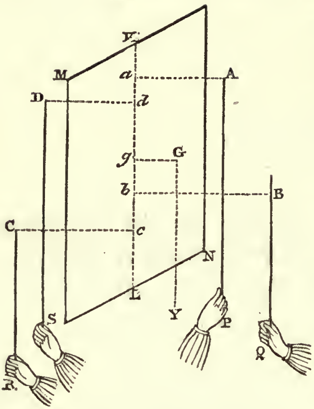
according as the forces are on the same side or on the opposite sides of the line KL. Hence we shall have, in the first case, *Fig.* 22,

$$\text{G}g=\frac{\text{P}\times\text{A}a+\text{Q}\times\text{B}b\ldots+(\text{R}\times\text{C}c+\text{S}\times\text{D}d\ldots)}{\text{P}+\text{Q}+\text{R}+\text{S}\ldots},$$

and, in the second case, *Fig.* 23,

$$\text{G}g=\frac{\text{P}\times\text{A}a+\text{Q}\times\text{B}b\ldots-(\text{R}\times\text{C}c+\text{S}\times\text{D}d\ldots)}{\text{P}+\text{Q}+\text{R}+\text{S}};$$

Fig. 23.



that is to say, when several parallel forces, situated in the same plane, act in the same direction, the distance of their resultant from any right line, traced in the same plane and parallel to their directions, is in general equal to the excess of the sum of the moments of the forces situated on one side of the line, over the sum of the moments of the forces situated on the other side, divided by the sum of the forces.

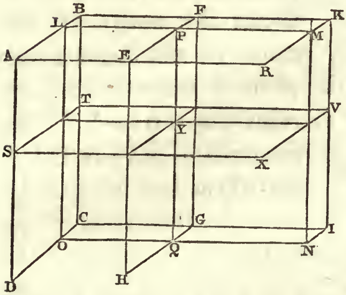
PROBLEM.

74. *An indefinite number of parallel forces being given, which act in the same direction and whose points of application may or may not be situated in the same plane, to determine, by means of moments, the direction of the resultant of all these forces.*

Solution. Having drawn at pleasure two different planes ABCD, BCIK, parallel to the directions of the forces, find the distance of the resultant from each of these planes separately (71); then draw a plane EFGH parallel to ABCD, at the distance from this latter plane that the resultant is from it, and situated on the side on which the sum of the moments referred to the plane ABCD is the greater; then this plane EFGH will contain

the direction required (71). Likewise draw a plane LMNO parallel to BCIK, at a distance from this latter plane equal to that of the resultant from it, and situated on the side on which the sum of the moments referred to the plane BCIK, is the greater; and this plane will also contain the direction required. Hence the direction of the resultant being both in the plane EFGH, and in the plane LMNO, it will be in the line of intersection PQ of these two planes.

Fig. 24.



COROLLARY I.

75. We have seen (30) that if several parallel forces change in direction, without changing either in intensity or their points of application, and without ceasing to be parallel to each other, their resultant always passes through the same definite point, which has been termed the centre of parallel forces; hence, for the parallel forces which we have just been considering, the centre is placed in the direction PQ of their resultant.

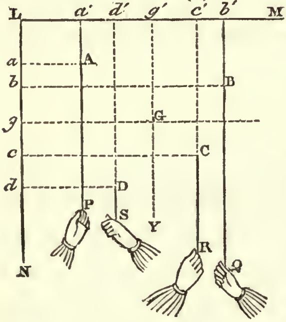
To find this centre, draw at pleasure a third plane ABKR (*Fig.* 24), and conceive that all the forces, without changing either their intensities or points of application, are directed parallel to each other and to the plane ABKR; find the distance of the resultant of these new forces from this plane (71). This done, if we draw a plane STVX parallel to ABKR, and at a distance from

this latter plane equal to that we shall have found, this plane will contain the new resultant, and consequently the centre of the forces. Hence the centre of the forces, being found both in the line PQ and in the plane STVX, it will be found in the point of intersection Y of the line and the plane; or, which is the same thing, this centre will be found at the point of intersection Y of the three planes EFGH, LMNO, STVX.

Corollary II.

76. If the parallel forces P, Q, R, S, . . . acting in the same direction, be situated in the same plane; in order to find the position of their resultant, draw in this plane a line LN parallel to the directions of the forces, and having let fall upon this line, from all the points of application A, B, C, D, . . . the perpendiculars Aa, Bb, Cc, Dd . . ., lay off upon a line LM perpendicular to LN, the line Lg', equal to (73)

Fig. 25. (a)



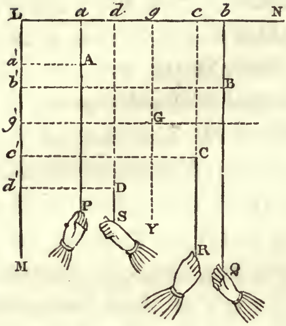
$$\frac{P\times Aa+Q\times Bb+R\times Cc+S\times Dd\ .\ .\ .}{P+Q+R+S\ .\ .\ .};$$

and the line g'Y, drawn through the point g' parallel to LN, will be the direction of the resultant.

If all the forces were not on the same side of the line LN, it would be necessary to subtract the moments of the forces situated on the other side, instead of adding them (73).

COROLLARY III.

Fig. 25. (*b*)



77. To find the centre of the parallel forces P, Q, R, S, . . . included in the same plane, conceive these forces, without changing their intensities and without ceasing to be applied to the same points A, B, C, D, . . . to be in directions parallel to another line, as LM, upon which let fall the perpendiculars Aa', Bb', Cc', Dd', ... and the distance Gg' of this line from the new resultant will be (73)

$$\frac{P \times Aa' + Q \times Bb' + R \times Cc' + S \times Dd' \ldots}{P + Q + R + S \ldots}.$$

Hence, if we lay off upon a line perpendicular to LM, the line Lg equal to this distance, and through the point g draw gG parallel to LM, this line gG will be the direction of the new resultant. Now, the centre of the forces is to be found both upon the direction of the first resultant g'Y, and upon that of the second gG; hence it will be at the point of intersection G of these two directions.

If all the forces were not on the same side of the line LM, it would be necessary to subtract the moments of those which are situated on the other side, instead of adding them.

COROLLARY IV.

78. If the points of application A, B, C, D, . . . (*Fig.* 25) are in the same plane to which the directions of the parallel forces P, Q, R, S, . . . are oblique, the centre G of these forces will also be in this plane (30), and its position will be the same as though the directions of the forces were parallel to each other and situated in this plane. Thus, to find in this case the centre of the forces G, draw in the plane any two right lines LN, LM; then suppose that the forces are in a direction parallel to LN, and find (77) the direction g'Y of their resultant on this supposition; then suppose they are in a direction parallel to LM, and find the direction Gg of their resultant; the point of intersection G of the two lines g'Y, gG will be the centre of the forces required.

The centre being found, if we draw through this point a line parallel to the real directions of the forces P, Q, R, S, . . . this line will be the direction of their resultant.

COROLLARY V.

79. Finally, if the points of application a', b', c', d', . . . (*Fig.* 25) be upon the same straight line LM oblique to the direction of the forces, the centre g' of these forces will be upon this line (30), and its position will be the same as though the directions of the forces were perpendicular to LM.

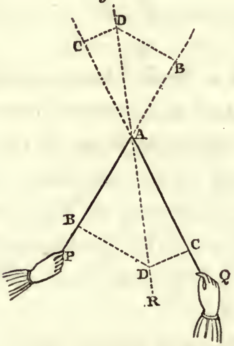
Hence (76) the distance g'L of this centre from a given point L upon the line, will be equal to

$$\frac{\text{P}\times a'\text{L}+\text{Q}\times b'\text{L}+\text{R}\times c'\text{L}+\text{S}\times d'\text{L}\ \ldots}{\text{P}+\text{Q}+\text{R}+\text{S}\ \ldots}.$$

If all the forces are not situated on the same side with reference to the point L, it will be necessary to subtract the moments of those which are situated on the other side, instead of adding them.

LEMMA.

Fig. 26.



80. *When the directions of two forces* P, Q, *meet in a point* A, *the moments of these forces, referred to any point* D *in the line of direction of their resultant* R, *are equal.*

For we have seen (35) that if from the point D the perpendiculars DB, DC, be dropped upon the directions of the forces, prolonged if necessary, we shall have

$$\text{P} : \text{Q} :: \text{DC} : \text{DB}.$$

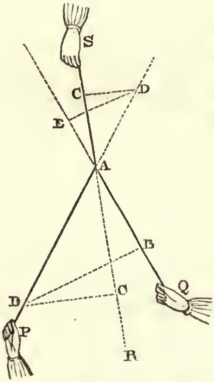
Hence, the product of the extremes being equal to that of the means, we shall have

$$\text{P}\times\text{DB}=\text{Q}\times\text{DC}.$$

COROLLARY.

81. From this it follows that, if the directions of the two forces P, Q meet in a point A, the moment of any one, Q, of these forces, referred to a point D in the direction of the other, will be equal to the moment of their resultant R referred to the same point: that is to say, by letting fall from the point D, the perpendiculars DB, DC upon the direction of the force Q, and upon that of the resultant R, prolonged if necessary, we shall have

Fig. 27.



$$Q \times DB = R \times DC.$$

For, by applying to the point A a third force S, equal and directly opposed to the resultant R, the three forces P, Q, S, will be in equilibrium; consequently the force P will be equal and directly opposed to the resultant of the two forces Q, S. Hence, the moments of the two forces Q, S, referred to the point D in the direction of their resultant, will be equal (80); hence we shall have

$$Q \times DB = S \times DC;$$

or, since $S = R$,

$$Q \times DB = R \times DC.$$

THEOREM.

Fig. 28.

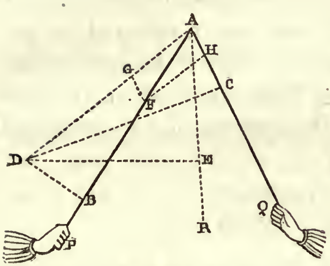
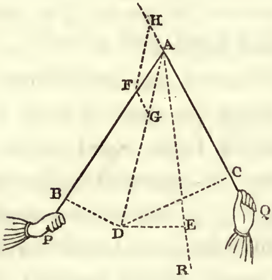


Fig. 29.



82. *When the directions of the two forces* P, Q, *meet in the same point* A, *the moment of the resultant* R *of these forces referred to any point* D, *taken in the plane of these directions, is equal to the sum or difference of the moments of the forces* P, Q, *referred to the same point, according as the point* D *is without or within the angle* PAQ, *formed by the directions of these forces: that is to say, if from the point* D *we let fall upon these directions, and upon that of the resultant, the perpendiculars* DB, DC, DE, *we have, in the first case,*

$$R \times DE = Q \times DC + P \times DB;$$

and, in the second case,

$$R \times DE = Q \times DC - P \times DB.$$

DEMONSTRATION. Draw the line AD, and decompose the force P into two others, p, p', the first in the direction AD, and the second in the direction of the force Q. For this purpose (38), represent the force P by the part AF of its direction; through the point F draw the lines FG, FH, respectively parallel to AQ and AD; and the two components p, p' will be represented by the sides AG, AH of the parallelogram AGFH.

The point D being upon the direction of the component p, the moment of the other component p', referred to this point, is equal to the moment of their resultant P, and we shall have (81)

$$p' \times \text{DC} = \text{P} \times \text{DB}.$$

Moreover, taking the two forces p, p' instead of the force P, the resultant R, of the two forces P, Q, is also the resultant of the three forces p, p', Q.

This being done, in the first case, the two forces Q and p' (*Fig.* 28), which act along the same line of direction, are equivalent to a single force equal to their sum $\text{Q}+p'$; thus the force R may be regarded as the resultant of the two forces p and $\text{Q}+p'$; hence the moment of this resultant referred to the point D, in the direction of the first of these forces, is equal to the moment of the second (81); hence we shall have

$$\text{R} \times \text{DE} = (\text{Q}+p')\ \text{DC},$$

or

$$\text{R} \times \text{DE} = \text{Q} \times \text{DC} + p' \times \text{DC}.$$

Hence, by substituting for the moment $p' \times \mathrm{DC}$ its value, we have

$$\mathrm{R} \times \mathrm{DE} = \mathrm{Q} \times \mathrm{DC} + \mathrm{P} \times \mathrm{DB}.$$

In the second case, the two forces Q, p' (*Fig.* 29), which are in the same line, and which act in contrary directions, are equivalent to a single force equal to their difference $\mathrm{Q} - p'$: now, the moment of this single force, referred to the point D in the direction of the force p, is equal to the moment of the resultant R of these two forces (81); and we have

$$\mathrm{R} \times \mathrm{DE} = (\mathrm{Q} - p')\mathrm{DC},$$

or

$$\mathrm{R} \times \mathrm{DE} = \mathrm{Q} \times \mathrm{DC} - p' \times \mathrm{DC}.$$

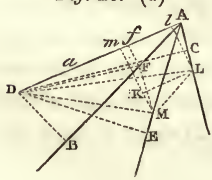
Hence, substituting for the moment $p' \times \mathrm{DC}$ its value, we have

$$\mathrm{R} \times \mathrm{DE} = \mathrm{Q} \times \mathrm{DC} - \mathrm{P} \times \mathrm{DB}.$$

Remark I.

83. This theorem (82) of Statics is a consequence of the following geometrical proposition:

Fig. 28. (*a*)



If from any point D, taken in the plane of a parallelogram AFML, we let fall the perpendiculars DB, DC, DE, upon the sides and the diagonal of this parallelogram, which meet in the same point A, the product

of the diagonal AM by its perpendicular DE is equal to the sum of the products of the sides multiplied each by its perpendicular DB and DC. Among the known demonstrations of this theorem, the following is one of the simplest: The triangles ADF, ADL, ADM, having the same base AD, are to each other as their altitudes Ff, Ll, Mm; but we have Mm=Ff+Ll; for drawing LK parallel to AD, we have Mm=MK+Km=Ff+Ll; hence the triangle ADM is equal to the sum of the triangles ADF and ADL: hence

$$\mathrm{AM}\times\mathrm{DE}=\mathrm{AF}\times\mathrm{DB}+\mathrm{AL}\times\mathrm{DC}.$$

Remark II.

Fig. 28.

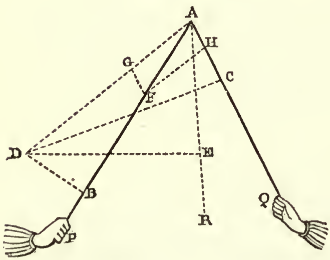
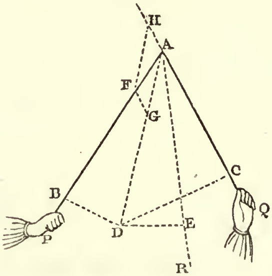


Fig. 29.



84. If we suppose the line AD to be inflexible and the point D immoveable; when this point is placed without the angle PAQ (*Fig.* 28), the two forces P, Q tend to turn the point A in the same direction around the point D; and, on the contrary, when the point D is placed within the angle PAQ (*Fig.* 29), the two forces tend to turn the point A in opposite directions.

Hence, if two forces be directed in the same plane, the moment of their resultant, referred to any point taken in this plane, is equal to the sum or difference of their moments referred to the same point, according as these forces tend to turn their point of application around the centre of moments, either in the same or in opposite directions; and, in all cases, the resultant tends to turn its point of application in the same direction as that of the two forces whose moment is the greater.

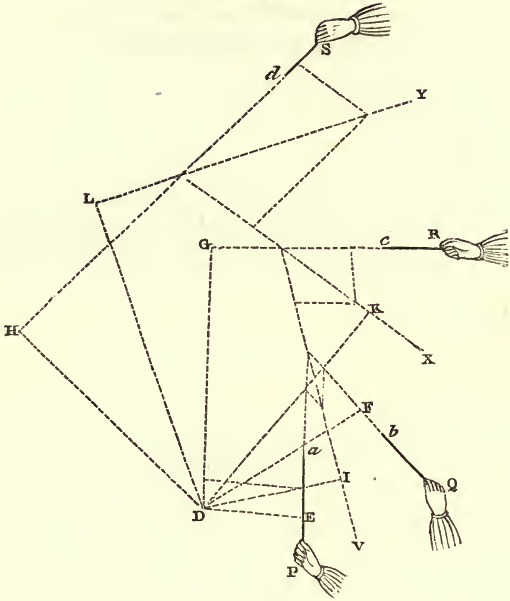
THEOREM.

85. *When the forces* P, Q, R, S . . . . . (*Fig.* 30), *directed in the same plane, are applied to the points a, b, c, d, connected together in an invariable manner, tending to turn these points in the same direction around another point* D *taken in this plane, the sum of the moments of these forces, referred to the point* D, *is equal to the moment of their resultant referred to the same point.*

DEMONSTRATION. Let V be the partial resultant of the two forces P, Q; X that of the two forces V, R, and consequently of the three forces P, Q, R; Y that of the two forces X, S, and consequently of the four forces P, Q, R, S; and so on. Then, from the point D let fall upon the directions of the forces and upon those of the partial resultants V, X, Y, . . . the perpendiculars DE, DF, DG, DH, ... DI, DK, DL, . . . This being done, the moment of the resultant V is equal to the sum of the moments of its components P, Q, (82), which gives

$$V \times DI = P \times DE + Q \times DF.$$

Fig. 30.



In like manner the moment of the resultant X is equal to the sum of the moments of its components V, R, and we have

$$X \times DK = V \times DI + R \times DG;$$

or, substituting for the moment $V \times DI$ its value,

$$X \times DK = P \times DE + Q \times DF + R \times DG.$$

Likewise, we have $Y \times DL = X \times DK + S \times DH$, or, substituting the value of $X \times DK$,

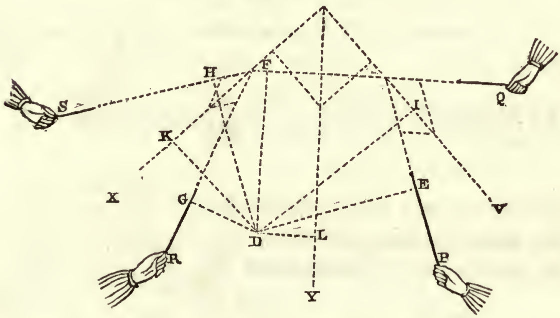
$$Y \times DL = P \times DE + Q \times DF + R \times DG + S \times DH;$$

and so on, whatever may be the number of forces. Hence the moment of each resultant is equal to the sum of the moments of all its components.

COROLLARY I.

86. *If the forces* P, Q, R, S . . . . *do not tend to turn their points of application in the same direction around the centre of the moments* D, *the moment of their resultant is equal to the excess of the sum of the moments of the forces which tend to turn in one direction, over the sum of the moments of those which tend to turn in the opposite direction.*

Fig. 31.



Thus, let V be the partial resultant of all the forces P, Q, . . . which tend to turn in one direction; and let X be the partial resultant of all the forces R, S, . . . which tend to turn in the opposite direction; from the point D let fall upon the directions of the forces and upon those of the two resultants V, X, the perpendiculars DE, DF, . . . DG, DH, . . . DI, DK; we have just seen (85), that we have

$$V \times DI = P \times DE + Q \times DF \ldots$$

and

$$X \times DK = R \times DG + S \times DH \ldots$$

Finally, let Y be the resultant of the two forces V, X, and consequently that of all the forces P, Q, R, S, . . .

This being established, the moment of the resultant Y referred to the point D is equal to the difference of the moments of its components V, X, which tend to turn in opposite directions (84): that is to say, by letting fall the perpendicular DL upon its direction, we have

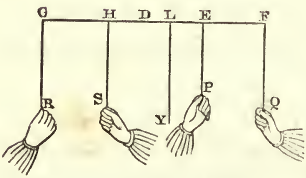
$$Y \times DL = V \times DI - X \times DK.$$

Hence, by substituting the values of the two moments, we have

$$Y \times DL = P \times DE + Q \times DF \ldots - (R \times DG + S \times DH \ldots).$$

Corollary II.

Fig. 32.



87. If the directions of the forces P, Q, R, S, . . . included in the same plane, are parallel to each other, the perpendiculars DE, DF, DG, DH, . . . DL let fall from the centre of moments D upon these directions and upon that of the resultant Y, will be in the same straight line; and the

preceding proportion will remain just the same, whether all the forces act in the same direction, as in *Fig.* 32, or whether they act, some in one direction and others in the opposite direction, as in *Figs.* 33 and 34. Now, the resultant Y of all these forces is equal to the excess of the sum of those which act in one direction, over the sum of those which act in the opposite direction (29); hence the distance DL of the centre of moments from the direction of the resultant, is equal to the quotient of the excess of the sum of the moments of the forces which tend to turn in one direction, over the sum of the moments of those which tend to turn in the opposite direction, divided by the excess of the forces which act in one direction, over the sum of those which act in the contrary direction: thus we have

Fig. 33.

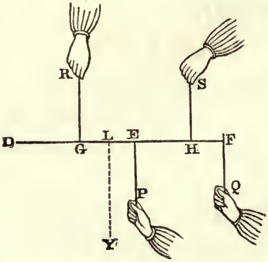
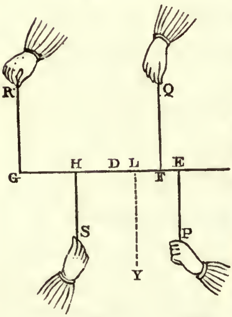


Fig. 34.



$$\mathrm{DL}=\frac{\mathrm{P}\times\mathrm{DE}+\mathrm{Q}\times\mathrm{DF}\ \ldots-(\mathrm{R}\times\mathrm{DG}+\mathrm{S}\times\mathrm{DH}\ \ldots)}{\mathrm{P}+\mathrm{Q}+\mathrm{R}+\mathrm{S}\ \ldots}, \quad (\textit{Fig. } 32.)$$

$$\mathrm{DL}=\frac{\mathrm{P}\times\mathrm{DE}+\mathrm{Q}\times\mathrm{DF}\ \ldots-(\mathrm{R}\times\mathrm{DG}+\mathrm{S}\times\mathrm{DH}\ \ldots)}{\mathrm{P}+\mathrm{Q}\ \ldots-(\mathrm{R}+\mathrm{S}\ \ldots)}, \quad (\textit{Fig. } 33.)$$

$$\mathrm{DL}=\frac{\mathrm{P}\times\mathrm{DE}+\mathrm{R}\times\mathrm{DG}\ \ldots-(\mathrm{Q}\times\mathrm{DF}+\mathrm{S}\times\mathrm{DH}\ \ldots)}{\mathrm{P}+\mathrm{S}\ \ldots-(\mathrm{Q}+\mathrm{R}\ \ldots)}, \quad (\textit{Fig. } 34.)$$

In all cases, the resultant acts in the direction in which the sum of the forces is the greater; and it is placed, with reference to the point D, on the side on which the sum of the moments is the greater.

CHAPTER THIRD.

ON CENTRES OF GRAVITY.

88. THE property by virtue of which bodies, left to themselves, fall towards the earth is named *gravitation* or *gravity*.

All the molecules, of which bodies are composed, have gravitation, and they always have it; for into whatever number of parts a body is divided, each of these parts continually gravitates, and falls towards the earth when left to itself.

89. The effort which a body makes to fall, when it is retained or supported by an obstacle which opposes its fall, is called the *weight* of the body; this weight may be regarded as the effect of a force which is constantly applied to the body: thus we are accustomed to consider gravity as a force.

Gravity is not a force rigorously constant for the same molecule; it varies according to the different positions which this molecule has relatively to the sphere of the earth.

1st. When the distance of the molecule from the centre of the earth changes, its gravity decreases in the same ratio as the square of this distance increases; besides, the earth not being perfectly spherical, and the lines drawn from its centre to the equator being greater

than those which terminate at the poles, the gravity at the surface of the earth is greater for the same molecule, when this molecule is placed at the poles, than when it is at the equator, because there the distance of the molecule from the centre of the earth is less.

2d. The earth turns around its axis, and all the parts composing it perform their revolutions in the same time, that is to say, in about twenty-four hours. The parts of the surface near the equator describe greater circumferences of circles than those described by the parts near the poles; their centrifugal force, which likewise is greater, destroys a greater part of the effect of gravitation, and is a new cause which renders this latter force less at the equator than it is at the poles.

Thus, rigorously speaking, gravity is variable for the same molecule, when this molecule departs from or approaches the surface of the earth, and when it departs from or approaches the equator: but the distances of the positions in which we are accustomed, in Statics, to consider the same molecule, are so small with reference to the radius of the earth, that the effects of this variation are absolutely insensible; and we may regard gravity as a constant force for the same molecule, whatever its position may be.

90. The straight line along which a molecule, abandoned to itself, falls to the earth, and which is evidently the direction of gravitation, is named a *vertical;* this line is everywhere perpendicular to the surface of the earth, or, more exactly, to the surface of undisturbed water.

91. A plane is said to be horizontal when it is perpendicular to a vertical.

If the earth were perfectly spherical all the lines of direction of gravitation would meet in the same point, which would be the centre: but the earth not being a perfect sphere, the lines of direction of gravitation for two different molecules may not be in the same plane; and when they are in the same plane, they meet in the same point.

However, the molecules in the same body, and those of the different bodies we are accustomed to consider in Statics, are so near each other compared with their distances from the centre of the earth, that the angle formed by the directions of gravitation for any two of them is not sensible, and we may regard all these directions as parallel.

92. We will regard, then, all the molecules of heavy bodies as constantly pushed or drawn towards the earth by forces constant for each of them; we will suppose that these forces are parallel, and act in the same direction; and, consequently, we will be able to apply to them all we have said of the composition, decomposition, and equilibrium of parallel forces.

Now, when several parallel forces, acting in the same direction, are applied to points invariably connected together, we have seen: 1st, that these forces have a resultant equal to their sum (27); 2d, that the direction of this resultant is that of the components; 3d, that there exists a centre of forces through which this resultant always passes, even though the forces, without changing in intensity and without ceasing to be parallel, should change in direction (30).

Hence, 1st, the weights of all the molecules of a solid body have a resultant which constitutes the weight of

the body, and this resultant is equal to the sum of the weights of the molecules; 2d, the direction of this resultant, or of the weight of the body, is always parallel to that of gravitation, and consequently vertical; 3d, whatever may be the different positions given to this body, the directions of the resultants for all these positions meet in the same point; for by varying the position of the body, the intensity of the forces, which act upon the molecules, is not altered, and these forces, which only change in direction with reference to the bodies, do not cease to be parallel to each other.

93. The point through which the direction of the weight of a body always passes, whatever may be its position, is named the *centre of gravity*.

94. When several bodies are invariably connected together, and we consider their assemblage as though they made but one and the same body, we ordinarily give to this assemblage the name of *system*.

95. Every thing that has just been said of a single body may likewise be said of a system of several bodies: that is to say, the weight of the system is equal to the sum of the partial weights of the bodies which compose it; that the direction of this weight is vertical, and that this direction, whatever may be the position of the system, always passes through the same definite point, which is the *centre of gravity of the system*.

Corollary I.

96. We may always regard the weight of a body, or system of several bodies, as a force directed verti-

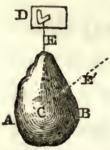
cally, and applied to the centre of gravity of the body or system: for this weight, which is the resultant of the partial weights of all the molecules which compose the body or system, may be considered as applied to any point of its direction, and consequently to the centre of gravity, which is always upon this direction, whatever may be otherwise the position of the body or the system.

Corollary II.

97. Hence, we will produce an equilibrium in the action which gravitation exerts upon all the molecules of a body or system of bodies, by applying to the centre of gravity of the body or system, a single force whose direction is vertical, equal to the total weight of the body or system, and which acts in a direction opposite to that of gravitation.

Inversely, when a single force produces an equilibrium in the weight of all the molecules of a body or system of bodies, the direction of this force will be vertical, and it will pass through the centre of gravity of the body or system.

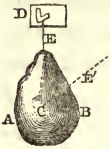
Fig. 35.



Thus, when a body AB, suspended by a thread ED from a fixed point D, is in equilibrium, and the action of gravitation is consequently destroyed by the resistance of the thread, the direction of this thread will be vertical, and its prolongation will pass through the centre of gravity C of the body.

Corollary III.

Fig. 35.



98. From this, we deduce a simple manner of finding, by experiment, the centre of gravity of a body of any figure. Thus, if we suspend the same body by a thread, successively by the two different points E, E′, and conceive the two directions of the thread to be prolonged into the interior of the body, the point C, in which these two directions intersect, will be the centre of gravity required.

Remark.

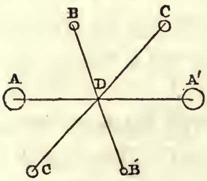
99. Since the partial weights of the bodies which compose a system, may be considered as parallel forces, applied to the partial centres of gravity of these bodies, it follows, that when we know the weight of these bodies, and the positions of their partial centres of gravity, we can find the position of the centre of gravity, by the processes which have been given to find the centres of parallel forces, either by means of the principle of the composition of parallel forces, as in No. 28, or by employing the consideration of moments, as in Nos. 75, 77, 78, and 79; we will soon have occasion to give examples.

To find the centre of gravity of a body of any figure whatever, conceive the body to be divided into a certain number of parts, so that we may know the weight of each of them, and the position of its partial centre of gravity; then, by finding the centre of gravity of the system of all these parts, we will have the required centre of gravity of the body.

But when the parts of the body are of the same nature in all its extent, and when the figure of the body is not very complicated, we may often find its centre of gravity by simpler considerations, and which we are about to employ in order to arrive at results which are used very frequently.

LEMMA.

Fig. 36.



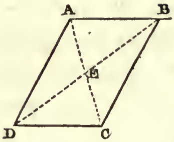
100. *When a body is considered to be composed of parts* A, A′, B, B′, C, C′, . . . *which, taken two and two, are equal to each other, and so placed that the middle of the lines* AA′, BB′, CC′, *which join the centres of gravity of the homologous parts, coincide in the same point* D, *this point, which is the centre of figure of the body, is also its centre of gravity.*

For the point D is the centre of gravity of each partial system of two homologous parts; hence, it is also the centre of gravity of their general system.

COROLLARY.

101. By considering lines, surfaces and solids, as composed of parts uniformly heavy, it is evident: 1st, that the centre of gravity of a right line is at the middle of its length;

Fig. 37.



2d. The centre of gravity of the area, and that of the contour of a parallelogram ABCD, are in its centre of figure: that is to say, at the point of intersection E, of its two diagonals AC, BD;

3d. The centre of gravity of the area of a circle, and that of its whole circumference, are at the centre of the circle;

4th. The centre of gravity of the whole surface of a parallelopipedon, and that of its solidity, are in its centre of figure: that is to say, in the intersection of any two of its four diagonals, or in the middle of one of them;

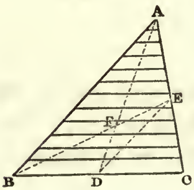
5th. The centre of gravity of the convex surface of a right or oblique cylinder, and that of its solidity, are in the middle of the length of its axis;

6th. The centre of gravity of the surface of a sphere, and that of its solidity, are at the centre of the sphere.

PROBLEM.

102. *To find the centre of gravity of the area of any rectilinear triangle* ABC.

Fig. 38.

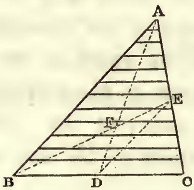


SOLUTION. Having drawn through the summit A of one of the angles, and through the middle D of the opposite side, the line AD, if we conceive the area of the triangle to be divided into an infinite number of elements by lines parallel to BC, the

centre of gravity of each of these elements will be in its middle (101), and consequently upon the line AD; hence the centre of gravity of their system, which will be that of the area of the triangle, will be upon this same line (30). For the same reason, if from the summit B of another angle, and through the middle E of the opposite side, we draw a line DE, this second line will contain the centre of gravity: hence this centre will be found both upon the line AD, and upon the line BE; hence it will be found at the point of intersection F of these two lines.

Corollary I.

Fig. 38.



103. *If from the summit* A *of one of the angles of the triangle* ABC, *and through the middle* D *of the opposite side, we draw a line* AD, *and divide this line into three equal parts, the centre of gravity* F *of the area of the triangle will be upon this line, at the distance of two-thirds from the summit of the angles, or one-third from the opposite side.*

For, if we draw the line DE, this line will be parallel to AB, because the sides BC, AC, are cut proportionally in D and E; and the triangles ABF, DEF will be similar, because their corresponding angles will be equal; hence we shall have

$$AF : FD :: AB : DE.$$

But the similar triangles ABC, EDC, give

$$AB : DE :: BC : DC, \text{ or } :: 2 : 1 \ (102).$$

Hence we shall have

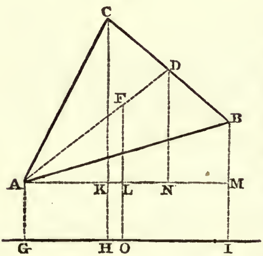
$$AF : FD :: 2 : 1, \text{ or } AF = 2FD,$$

hence,

$$FD = \tfrac{1}{3}AD, \text{ and } AF = \tfrac{2}{3}AD.$$

Corollary II.

Fig. 39.



104. *If in the plane of a rectilinear triangle* ABC *we draw any line* GI, *the perpendicular let fall from the centre of gravity* F *of the area of the triangle upon* GI, *will be equal to one-third of the sum of the perpendiculars* $AG+CH+BI$ *let fall from the summits of the angles upon the same line.*

Thus, through the summit A of one of the angles draw the line AM parallel to GI, which will cut in K, L, M, the perpendiculars let fall from the other points; through the point A, and through the centre of gravity F, draw the line AF, whose prolongation will bisect the opposite side at the point D; finally, through the point D, draw DN perpendicular to AM: this being done, we shall have

$$DN = \frac{CK+BM}{2},$$

and the similar triangles AFL, ADN will give

$$\text{FL} : \text{DN} :: \text{AF} : \text{AD}, \text{ or } :: 2 : 3 \ (103).$$

Hence,

$$\text{FL} = \tfrac{2}{3}\text{DN} = \frac{\text{CK} + \text{BM}}{3}.$$

But the lines AG, KH, LO, MI, being equal to each other, we shall have

$$\text{LO} = \frac{\text{AG} + \text{KH} + \text{MI}}{3}.$$

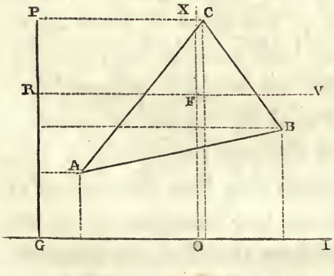
Hence, by adding this equation to the preceding, we shall have

$$\text{FL} + \text{LO} = \frac{\text{AG} + \text{CK} + \text{KH} + \text{BM} + \text{MI}}{3}:$$

that is to say,

$$\text{FO} = \frac{\text{AG} + \text{CH} + \text{BI}}{3}.$$

Fig. 40.



From this we deduce another manner of finding the centre of gravity of the area of a rectilinear triangle ABC. Having drawn at pleasure, in the plane of the triangle, two lines GI, GP, and having found the distances FO, FR, of the centre of gravity from each of these lines,

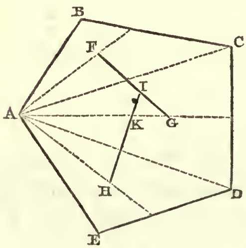
if we draw the line RV parallel to GI and at the distance FO, this line will contain the required centre of gravity; in like manner if we draw XO parallel to PG and at the distance FR, this second line will contain the centre of gravity; hence this centre will be found upon the two lines RV, XO; hence it will be at their point of intersection F.

PROBLEM.

105. *To find the centre of gravity of the area of a rectilinear polygon* ABCDE *of any number of sides.*

First Solution, *by the process of the composition of parallel forces.*

Fig. 41.



Divide the area of the polygon into triangles by the diagonals AC, AD, . . . drawn from the summit of the same angle A, and determine (102, or 103, or 104), the partial centres of gravity F, G, H of the areas of these triangles; then considering these triangles as weights proportional to their areas and applied to their centres of gravity, join the centres of gravity of the first two triangles ABC, CAD by a line FG, and find upon this line the centre of gravity I of the system of the two triangles, or of the quadrilateral ABCD, by dividing the line FG into two parts reciprocally proportional to the areas of the two

triangles (18,) which may be done by the following proportion (25):

quadrilateral ABCD : triangle CAD : : FG : FI.

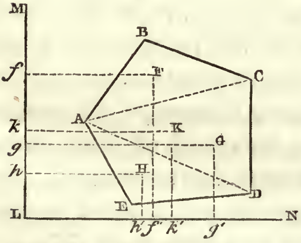
Through the point I, and through the centre of gravity H of the next triangle, draw the line HI, upon which find the centre of gravity K of the system of the first three triangles, by dividing this line into two parts reciprocally proportional to the areas of the quadrilateral ABCD and of the triangle DAE, which may be done by the following proportion:

pentagon ABCD : triangle DAE : : IH : IK.

By thus continuing, whatever may be the number of triangles, we will find the centre of gravity of their system, and this centre will be that of the area of the proposed polygon.

SECOND SOLUTION, *taken from the consideration of moments.*

Fig. 42.



Having divided the area of the polygon, as in the preceding solution, and determined the partial centres of gravity F, G, H, . . of all the triangles, draw at pleasure in the plane of the polygon two lines LM, LN, upon which let fall perpendiculars from all the centres of gravity F, G, H, . . ; consider these lines as the intersection of two planes paral-

lel to the direction of gravitation. This being done, the distance of the centre of gravity of the polygon, or of the system of all the triangles, from each of the lines LM, LN, will be equal to the sum of the moments of the triangles referred to each plane, divided by the sum of their areas (77): thus the distance of this centre from the line LM will be

$$\frac{\text{ABC}\times\text{F}f\pm\text{CAD}\times\text{G}g\pm\text{DAE}\times\text{H}h}{\text{ABCDE}},$$

and its distance from the line LN will be

$$\frac{\text{ABC}\times\text{F}f'\pm\text{CAD}\times\text{G}g'\pm\text{DAE}\times\text{H}h'}{\text{ABCDE}}.$$

Hence, by drawing a line parallel to LM, and at a distance equal to the first of these two distances, this line will contain the centre of gravity of the polygon; likewise if we draw a line parallel to LN, at a distance equal to the second of these distances, this line will contain the centre of gravity; hence the intersection of these two lines will be the required centre of gravity.

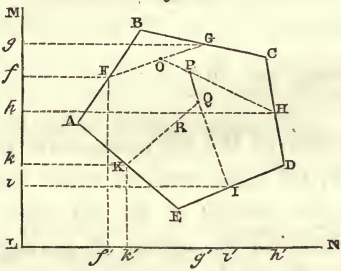
106. If the centres of gravity F, G, H, of the triangles which compose the area of the polygon, were not all placed on the same side with reference to each of the lines LM, LN, in order to find the distance of the centre of gravity K of the polygon from each of these lines, it would be necessary to subtract the moments of the triangles, whose centres of gravity are situated on the other side of this line, instead of adding them (77).

PROBLEM.

107. *To find the centre of gravity of the contour of a polygon* ABCDE, *of any number of sides.*

First Solution, *by the process of the composition of parallel forces.*

Fig. 43.



Bisect each of the sides of the polygon at the points F, G, H, I, K, which will be the partial centres of gravity of these sides (101). Then, considering all the sides as weights proportional to their lengths, find the centre of gravity of the system O of any two of them, as AB, BC, by joining their centres of gravity by the line FG, and dividing this line into two parts reciprocally proportional to these sides, which may be done by the following proportion (22):

$$AB+BC : BC :: FG : FO.$$

The point O being found, draw through this point, and through the middle H of the next side, the line OH, upon which find the centre of gravity P of the system of the three sides, by dividing this line into two parts reciprocally proportional to the side CD and the sum of the first two, AB, BC; which may be done by the proportion,

$$AB+BC+CD : CD :: OH : OP.$$

In like manner, drawing the line PI, find the centre of gravity Q of the system of the four sides AB, BC, CD, DE by the proportion,

$$AB+BC+CD+DE : DE :: PI : PQ.$$

By thus continuing, whatever may be the number of sides of the polygon, find the centre of gravity of their system, and this centre will be that of the contour of the polygon.

SECOND SOLUTION, *taken from the consideration of moments.*

Having bisected each side of the polygon, draw at pleasure the two lines LM, LN, upon each of which let fall perpendiculars from the middle of all the sides. This being done, the distance of the centre of gravity R of the system of all the sides referred to each of the planes, whose lines of intersection are LM, LN, will be equal to the sum of the moments of the sides referred to this plane, divided by the sum of the sides (77); thus, the distance of this centre from the line LM will be

$$\frac{AB\times Ff+BC\times Gg+CD\times Hh+DE\times Ii+EA\times Kk}{AB+BC+CD+DE+EA},$$

and its distance from the line LN will be

$$\frac{AB\times Ff'+BC\times Gg'+CD\times Hh'+DE\times Ii'+EA\times Kk}{AB+BC+CD+DE+EA}.$$

Hence, drawing a line parallel to LM, at a distance equal to the first of these distances, then another line parallel to LN, at a distance equal to the second of these

distances, the point of intersection of these two lines will be the centre of gravity R of the contour of the polygon.

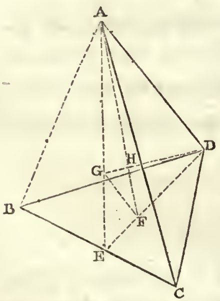
Remark.

108. If the middle points F, G, H, I, K (*Fig.* 43), of the sides of the polygon, were placed on opposite sides of the lines LM, LN; in order to find the distance of the centre of gravity R from each of these lines, it would be necessary to subtract the moments of the sides whose middle points were situated on the contrary side, instead of adding them (77).

PROBLEM.

109. *To find the centre of gravity of the solidity of any triangular pyramid* ABCD.

Fig. 44.



SOLUTION. Determine the centre of gravity F of the area of one of the faces BCD of the pyramid (103), by drawing through the summit D of one of the angles of this face, and through the middle E of the opposite side BC, a line DE; and take upon this line a point F two-thirds of the distance from the summit of the angle or one-third from the base; then draw the line AF. This being done, if we conceive the pyramid to be divided into an infinite number of sections by planes parallel to the face DCB, all

these sections will be similar to this face, and they will be met by the line AF in points, which, being situated upon each of them in the same manner as the point F is in the face BCD, will be the partial centres of gravity of these sections; hence the centre of gravity of their system, which will be that of the solidity of the pyramid, will be upon the line AF (30).

For the same reason, having determined the centre of gravity G of the area of another face ABC, which is done by drawing the line AE, and taking upon this line the part EG$=\frac{1}{3}$ AE; if through this point, and through the summit D of the opposite angle of the pyramid, the line DG be drawn, this line will also contain the centre of gravity of the solidity of the pyramid.

Hence the lines AF, DG, both containing the centre of gravity of the pyramid, will necessarily intersect in a certain point H; and the point of intersection of these two lines will be the centre of gravity required.

Remark I.

110. It might be demonstrated, independently of the consideration of the centre of gravity of the pyramid, that the lines AF, DG necessarily intersect in one point; for these lines are in the same plane, which is that of the triangle ADE.

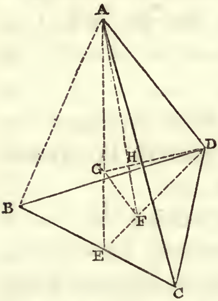
Remark II.

111. Any one of the six edges of a triangular pyramid being cut by four others; the fifth which does not meet it is called its *opposite:* if we join the middle point of one of the six edges with that of its opposite

by a line, it may be demonstrated that the middle of this line is the centre of gravity of the pyramid. (See the *Correspondence of the Polytechnic School*, tome II, page 1.)

COROLLARY I.

Fig. 44.



112. *If from the summit* A *of one of the angles of a triangular pyramid, and through the centre of gravity* F *of the area of the opposite face* BCD, *a line* AF *be drawn, the centre of gravity* H *of the solidity of the pyramid will be upon this line, and at one-fourth of the distance from the face, or at three-fourths of the distance from the summit of the angle.*

Draw the line GF, which will be parallel to AD, because the lines EA, ED, are cut proportionally in G, F; the triangles AHD, FHG, whose corresponding angles are equal, will be similar, and will give

$$AH : HF :: AD : GF.$$

But the similar triangles AED, GEF, give

$$AD : GF :: ED : EF, \text{ or } :: 3 : 1 \ (103);$$

hence, we shall have

$$AH : HF :: 3 : 1;$$

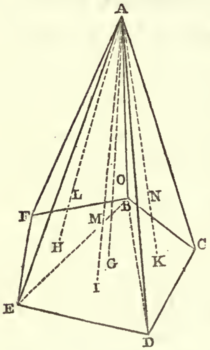
that is to say, $AH = 3HF$, and consequently $HF = \frac{1}{4}AF$, and $AH = \frac{3}{4}AF$.

COROLLARY II.

113. It might be demonstrated, in a manner analogous to that of No. 104, that the distance of the centre of gravity of the solidity of a triangular pyramid from any plane, is equal to the fourth of the sum of the distances of the summits of the four angles of the pyramid from the same plane.

COROLLARY III.

Fig. 45.



114. *The centre of gravity* O *of the solidity of a pyramid* ABCDEF *with any base is upon the line* AG, *drawn from the summit* A *to the centre of gravity* G *of the area of the base, and at a distance of one-fourth of this line from the base or three-fourths from the summit.*

Conceive the pyramid to be divided into an infinite number of sections by planes parallel to the base: all these sections will be similar to the base, and the point where each of them is intersected by the line AG will be situated upon this section in the same manner as the point G is upon the base; consequently this point will be the centre of gravity of the section: hence the centres of gravity of all the sections will be upon the line AG; hence the centre of gravity of their system,

which is that of the solidity of the pyramid, will be also upon this line (30).

Moreover, let the base be divided into triangles by the diagonals BE, BD, and conceive that through these diagonals and through the summit A, the planes ABE, ABD be drawn, which will divide the proposed pyramid into as many triangular pyramids as there are triangles in the base; then through the centres of gravity H, I, K of the triangular bases, draw the lines AH, AI, AK; finally, let the points L, M, N be taken upon these lines, upon each of them at the distance of one-fourth of its length from the base; these points will be the centres of gravity of the triangular pyramids (112). This being done, the points L, M, N, which will divide proportionally the lines AH, AI, AK, drawn from the summit of the pyramid upon the base, will be in the same plane parallel to the base; hence the centre of gravity of the system of triangular pyramids,—that is to say, the centre of gravity of the solidity of the proposed pyramid,—will be in this same plane; hence the centre of gravity, being found both in this plane and in the line AG, will be at the point of their intersection O.

Now, the line AG will be cut by the plane LMN in parts proportional to the divisions of the lines AH, AI, AK; hence the centre of gravity O of the solidity of the pyramid will be placed upon AG, at one-fourth of this line from the base, or three-fourths from the summit.

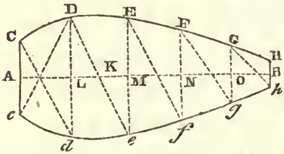
Corollary IV.

115. The centre of gravity of the solidity of a cone of any base is upon the line drawn from the summit to the centre of gravity of the base, and at the distance of one-fourth of this line from the base, or at three-fourths from the summit; for this solid may be considered a pyramid whose base has an infinite number of sides.

PROBLEM.

116. *To find the centre of gravity of the area of a section made in the hull of a vessel by a horizontal plane.*

Fig. 46.



Solution. Let CEH*hec* be the proposed section, AB the line of intersection of the plane of this section with the vertical plane drawn through the keel of the vessel. It is evident, since the whole section is symmetrical on each side of the line AB, the required centre of gravity K will be upon this line; thus, to construct this point, it will suffice to know its distance AK from a line C*c*, drawn through a given point perpendicular to AB.

For this purpose, let the line AB be divided by the perpendiculars or ordinates D*d*, E*e*, F*f*, . . . into a sufficiently great number of equal parts, so that the arcs CD, DE, EF, ... included between two adjoining perpendi-

culars, may be regarded as right lines, which will divide the area of the section into trapeziums; then let each of these trapeziums be divided into triangles, by means of the diagonals Cd, De, Ef, ... This being done, if we take the sum of the moments of all the triangles referred to the vertical plane passing through the line Cc, and divide this sum by the sum of the areas of the triangles, the quotient will be the distance required AK (77). Now, each triangle may be considered as having for base one of the perpendiculars, and for height the common distance from each other of two consecutive perpendiculars; hence the area of each triangle will be equal to the half of the product of the ordinate which serves for base, multiplied by the common distance. For example, the area of the triangle DEe will be equal to the half of the product Ee×LM; that of the triangle Dde will be the half of Dd×LM, and so of the others. Moreover, the distance of the centre of gravity of each of the triangles from the plane Cc, will be equal to one-third of the sum of the distances of the summits of its three angles from the same plane (104): for example, the distance of the centre of gravity of the triangle DEe from the plane Cc will be one-third of AL+AM+AM, and so of the others.

Hence it will be easy to have the sum of the areas of all the triangles, and the sum of the moments of these areas referred to the plane Cc; and by dividing the second of these two sums by the first, we shall have the required distance of the centre of gravity K from the line Cc.

The preceding solution is not rigorous, because the parts CD, DE, . . ., cd, de, . . ., of the sides of the sec-

tion are not right lines, as we have supposed; but it is evident that the result will approach exactness as much more as these parts are smaller: that is to say, as the number of perpendiculars are greater.

117. The operation just described is susceptible of some reduction. Thus, according to the preceding, the area of the triangle

$$\mathrm{C}cd = \mathrm{AL} \times \frac{\mathrm{C}c}{2},$$

$$\text{That of } \mathrm{CD}d = \mathrm{AL} \times \frac{\mathrm{D}d}{2},$$

$$\text{That of } \mathrm{D}de = \mathrm{AL} \times \frac{\mathrm{D}d}{2},$$

$$\text{That of } \mathrm{DE}e = \mathrm{AL} \times \frac{\mathrm{E}e}{2},$$

$$\text{That of } \mathrm{E}ef = \mathrm{AL} \times \frac{\mathrm{E}e}{2},$$

$$\text{That of } \mathrm{EF}f = \mathrm{AL} \times \frac{\mathrm{F}f}{2};$$

and so on with the others. By adding all these products together, we see that their sum is equal to the product of the common factor AL, multiplied by half the sum of the two extreme perpendiculars and the sum of all the others.

As to the moments of these triangles referred to the plane Cc, we have

$$\text{That of C}cd = \text{AL} \times \frac{\text{C}c}{2} \times \frac{1\text{AL}}{3},$$

$$\text{That of CD}d = \text{AL} \times \frac{\text{D}d}{2} \times \frac{2\text{AL}}{3},$$

$$\text{That of D}de = \text{AL} \times \frac{\text{D}d}{2} \times \frac{4\text{AL}}{3},$$

$$\text{That of DE}e = \text{AL} \times \frac{\text{E}e}{2} \times \frac{5\text{AL}}{3},$$

$$\text{That of E}ef = \text{AL} \times \frac{\text{E}e}{2} \times \frac{7\text{AL}}{3},$$

$$\text{That of EF}f = \text{AL} \times \frac{\text{F}f}{2} \times \frac{8\text{AL}}{3},$$

and so on; in which we see that the number which multiplies AL, in the moment of the last triangle, is always equal to three times the number of the intervals minus unity; or, which is the same thing, to three times the number of the last perpendicular, less 4. By adding together all these moments, we find their sum equal to the product of the common factor AL×AL, multiplied by the sum composed of one-sixth of the first perpendicular, one sixth of the last, multiplied by three times the number of perpendiculars less 4, then of the second perpendicular, double the third, three times the fourth, . . . and so on.

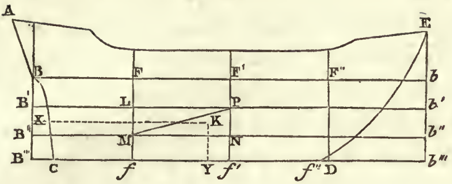
Now the sum of the moments and that of the areas having the common factor AL, their quotient will also be the same if we suppose this factor to be in both terms of the division; hence, to obtain the distance of the centre of gravity K from one of the extreme ordinates Cc, *it is necessary*, 1st, *to take one-sixth of the first ordinate* Cc *; one-sixth of the last* H*h*, *multiplied by three times the number of ordinates, less* 4 *; then the second ordinate, double the third, three times the fourth, . . . and so on ; which will give the first sum :* 2d, *to the half of the two extreme ordinates add all the intermediate ordinates ; this will give the second sum :* 3d, *divide the first of these two sums by the second, and multiply the quotient by the common interval of the ordinates.*

PROBLEM.

118. *To find the centre of gravity of the volume of the submerged part of the hull of a vessel.*

SOLUTION. We will suppose that the vessel, being afloat, has its keel horizontal, and that the vertical plane drawn through the keel divides the volume of the hull into two perfectly symmetrical parts. This being done, the centre of gravity of the part submerged will be in this plane, and the question will be reduced to find the distances of this point from two lines of known position in the vertical plane.

Fig. 47.



Let ABCDF be the section of a vessel through the vertical plane CD, its keel, and conceive the plane of flotation, or the section made in the vessel at the level of the water, to be represented by the line Bb parallel to the keel. Let the interval of the two lines Bb, CD, be divided into a certain number of equal parts BB$'$, B$'$B$''$, B$''$B$'''$ . . . ., and through each point of division suppose there are horizontal sections represented by B$'b'$, B$''b''$. . . . . In like manner let the line Bb, from the point B of the stern-post, be divided into equal parts BF, FF$'$, F$'$F$''$. . . . ; and through each point of division imagine vertical planes to be arranged perpendicular to the keel, and represented by the lines BB$'''$, Ff, F$'f'$. . . . ; the submerged part of the hull will be divided into rectangular prisms, whose sides will be perpendicular to the vertical plane drawn through the keel, and which will be terminated on both sides at the surface of the vessel. (It is necessary that the divisions of the lines BB$'''$, Bb should be so small that the part of the surface of the vessel, which terminates each prism, may be regarded as a plane). Finally, let each rectangular prism, represented by its base LMNP, be divided into two triangular prisms, by a diagonal plane, represented by MP.

This being done, 1st, each triangular prism will always be divided into three pyramids of the same base as the

prism (Legendre's Geometry, Book VI.), and each of which will have for height one of the sides of the prism: hence, if, by actual measurement of the vessel, we get the length of all the sides, it will be easy to find the solidity of each pyramid, by multiplying the area of the common base, LMP, by one-third of the side, which measures the height of the pyramid; and by taking the sum of all these solidities, we will obtain that of the submerged part of the hull. 2d. The moment of a triangular pyramid referred to a plane, being equal to the product of the solidity of the pyramid, multiplied by one-fourth of the sum of the distances of the summits of its four angles from this plane (113), it will be easy to find the moment of each pyramid referred to the vertical plane BB''' or to the horizontal plane CD; because the distances of the summits of these angles from each of these planes are known; and by taking the sum of all these moments, we shall have the moment of the submerged part of the hull.

This being done, the quotient of the sum of the moments referred to the vertical plane BB''', divided by the sum of the solidities, will be the distance KX of the required centre of gravity from the vertical BB'''; in like manner the quotient of the sum of the moments referred to the horizontal plane CD, divided by the sum of the solidities, will be the distance KY of the same point from the keel. We shall have, therefore, the distances of the centre of gravity from two lines of known position in the vertical plane drawn through the keel, and consequently the position of this point will be determined.

The preceding solution is not rigorous; because the surface of the vessel being curved, the part of this surface which terminates each triangular prism cannot be regarded as a plane, as we have supposed; but the result will approach exactness as much more as the number of divisions, both in the direction of the height of the vessel, and in that of its length, are greater.

119. The operation just described is susceptible of some reduction; and by reasoning as in No. 116 we find, that to get the distance KX of the centre of gravity of the submerged part of the hull from the vertical plane BB‴, *it is necessary*, 1st, *for each horizontal section, to take one-sixth of the first ordinate which is in the plane* BB‴, *one-sixth of the last, multiplied by three times the number of ordinates contained in the section, less* 4; *then the second ordinate, double the third, three times the fourth, . . ., which will form a partial sum for each section; then add together the half of the first of these sums, the half of the last, and all the intermediate ones, which will form a dividend;* 2d, *to one-fourth of the four ordinates placed at the angles of the rectangle* B$b$$b$‴B‴, *add one-half of all which are upon the contour of this rectangle, and the whole of all those in the interior, which will form a divisor;* 3d, *divide the dividend by the divisor, and multiply the quotient by the interval* BF *parallel to the distance required* KX.

To find the distance KY of the centre of gravity from the horizontal plane drawn through the keel, it is necessary to operate upon the vertical sections as upon the horizontal sections in the preceding case: that is to say, 1st, *for each vertical section, take one-sixth of the lower ordinates, one-sixth of that which is in the plane*

of flotation, multiplied by three times the number of ordinates of the section, less 4*: then the second ordinate from the bottom, double the third, three times the fourth, . . . which will form for each section a partial sum; then add together the half of the first of these sums, the half of the last, and all the intermediate ones, which will form a dividend;* 2d, *divide this dividend by the same divisor as in the preceding case, and multiply the quotient by the interval* BB *parallel to the distance sought* KY.

Remark.

120. In the preceding problem, the only object is to find the centre of gravity of the volume of the submerged part of the hull, or, which is the same thing, the volume of water displaced by the vessel. But if it were required to find the centre of gravity of the vessel itself either laden or unladen: that is to say, to find the distances of this point from the horizontal plane drawn through the keel, and from the vertical plane perpendicular to the keel, it would be necessary to take, with reference to each of these planes, the sum of the moments of all the parts which compose the vessel and its load, and then to divide each of these sums by the total weight of the vessel and its load; observing, in taking the moments, to multiply, not the volume, but the weight of each part, by the distance of the partial centre of gravity of this part from the plane to which the moments are referred; and the quotients of these divisions would be the distances required.

It will be easy to find, at least in a manner sufficiently near, the partial centre of gravity of each of the parts of the vessel and its load ; because we may always decompose this part into parallelopipedons, cylinders, pyramids, or other solids of which we have given the means of finding the centres of gravity.

CHAPTER FOURTH.

ON THE EQUILIBRIUM OF MACHINES.

121. Every instrument intended to transmit the action of a determined force, to a point which is not found upon its direction, so that this force may move a body to which it is not immediately applied, and move it in a direction different from its own, is called a *machine*.

122. In general, we are not able to change the direction of a force, but by decomposing this force into two others; one of which is directed toward a fixed point, which destroys the force by its resistance, and the other acts in a new direction: this latter force, which is the only one that can produce any effect, is always a component of the first; and may be either smaller or greater than it is, according to circumstances. By changing, in this manner, the directions and intensities of the forces, we may, by the aid of a machine and the points of support which it presents, produce an equilibrium between two unequal forces which are not directly opposed.

123. The force, whose direction is to be changed by employing a machine, is ordinarily named a *power*, and the term *resistance* is applied to the body it has to

move, or to the force with which it has to produce an equilibrium by means of the machine.

124. We propose here to find only the ratios which should subsist between the power and the resistance applied to the same machine, in order that, with regard to their directions, they may be in equilibrium. We will leave friction out of consideration: that is to say, the difficulties which the different parts of the machine may experience in slipping or rolling upon each other; and we will suppose that cords, when they enter into the composition of the machine, are perfectly flexible. Thus, having given to a power the intensity which is proper for the condition of equilibrium in this supposition, it would not suffice to augment it by a small quantity to destroy the equilibrium and put the machine in motion; but we must first apply the whole quantity required to overcome the obstacles just mentioned; and then a slight increase will produce motion.

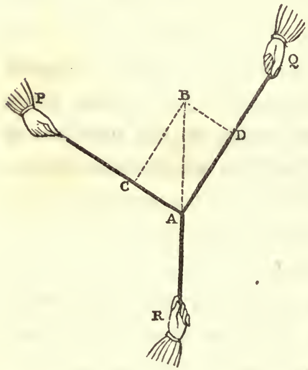
125. Although the number of machines is very great, we may regard them all as composed of three simple machines, which are: *cords*, the *lever*, and the *inclined plane*. We will content ourselves with presenting the theories of these three machines, and of those which are immediately derived from them; it will then be easy, by simple applications, to find the ratio of the power to the resistance, for the condition of equilibrium, in every machine, however complicated it may be.

Article I.

On the equilibrium of forces which act upon each other by means of cords.

126. We will suppose that the cords are without weight; and because the property they have of transmitting force, is independent of their size, we will suppose them to be reduced to their axes, and regard them as straight, flexible, and inextensible lines. Taking this for granted, let us consider, first, the case of equilibrium between three forces P, Q, R, acting upon each other by means of three cords united together by a knot A.

Fig. 48.



1st. The three forces P, Q, R, cannot be in equilibrium, unless the three directions, and consequently the cords by means of which they transmit their actions, are in the same plane (10).

2d. If we represent any two of these forces, for example, the forces P, Q, by the parts AC, AD of their directions, and upon these lines as adjacent sides construct the parallelogram ACBD, the diagonal AB will represent in intensity and direction the resultant of these two forces (36); now the three forces being in equilibrium, the force R must be equal and directly opposed to the resultant of the two others; hence the direction of the

force R will be in the prolongation of BA, and its intensity will be represented by this diagonal: thus we shall have

P : Q : R : : AC : AD : AB;

or because the sides AD, BC of the parallelogram are equal to each other, the three forces P, Q, R will be to each other as the sides of the triangle ABC.

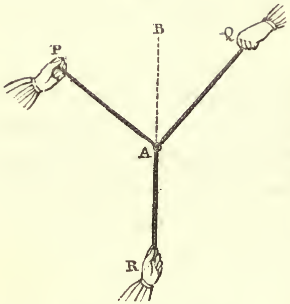
127. The angles of the triangle ABC, being given by the directions of the forces P, Q, R, and the magnitudes of its sides being proportional to the intensities of these forces, it follows that, of the six things to be considered in the equilibrium in question, namely, the directions of the forces and their intensities, any three being given, we can find the three others in all cases in which, of the six things to be considered in the triangle ABC, namely, the angles and the sides, three being given, we can determine the three others.

For example, when the three forces P, Q, R are known, we can find the angles which the cords make with each other when in equilibrium, by constructing the triangle ABC, the sides of which are proportional to these forces. But when the directions are given, we can know only the ratios of the three forces; because in the triangle ABC, the knowledge of the three angles determines only the ratios of the sides, but does not determine their magnitudes. Thus, it will be necessary, besides, to know the magnitude of one of the three sides P, Q, R, in order to find that of the two others, by means of the proportional series:

P : Q : R : : AC : BC : AB.

Remark I.

Fig. 49.



128. If the three cords be united by a slip-knot; for example, if the cord PAQ passes through a ring attached to the extremity of the cord RA, the conditions just enunciated are not sufficient to establish equilibrium: it is necessary, in addition, that the angles PAB, QAB, formed by the two parts of the cord and by the prolongation AB of the direction of the other cord, should be equal; for it is evident that, otherwise, the ring A would slip upon this cord toward the greater of the two angles.

Remark II.

129. What has just been said contains the whole theory of equilibrium between three powers applied to cords united in the same knot; but we have supposed the construction of the parallelogram ACBD, we may however enunciate this theory independently of all construction.

Thus, in every triangle ABC, the sides are proportional to the sines of the opposite angles: that is to say, we have

$$AC : BC : AB :: \sin ABC : \sin BAC : \sin ACB.$$

Now, the sines of these angles are respectively the same as those of their supplements RAQ, RAP, PAQ; hence we have also

AC : BC : AB :: sin RAQ : sin RAP : sin PAQ,

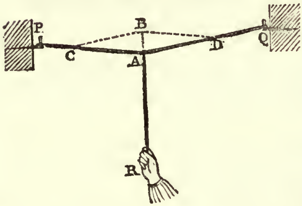
and consequently

P : Q : R :: sin RAQ : sin RAP : sin PAQ.

That is to say, *when the three powers, which act by cords upon the same knot, are in equilibrium, each of them is as the sine of the angle formed by the directions of the other two.*

Corollary I.

Fig. 50.



130. If the cords AP, AQ, instead of being drawn by two powers, be attached to two fixed points at P and Q, and we represent the force R by the diagonal AB of the parallelogram ABCD, the two sides AC, AD will represent the tensions of these two cords, or the efforts which they exert upon the fixed points in the line of their directions.

Corollary II.

131. When the angle PAQ is very large, the sides AC, AD of the parallelogram are very great compared with the diagonal AB, and consequently the effort which the

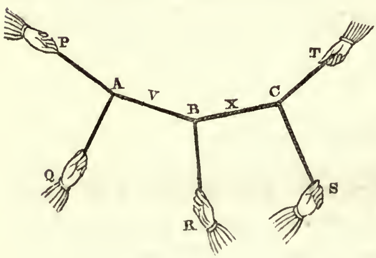
power R exerts upon the fixed points P, Q, to make them approach each other, is very great compared with this power. Hence, by means of cords, we can put a moderate power in the condition of producing a very great effect.

Corollary III.

132. In the case of equilibrium, however small the force R may be, the diagonal AB, which represents it, is not zero, and the three points C, A, D, are not in a straight line; hence, by supposing that a cord PAQ without weight is stretched in a line by two forces P, Q, the smallest force R, applied to A, will bend it at this point and cause it to make an angle PAQ. Thus, it is rigorously impossible to stretch a heavy cord in a straight line, unless it be vertical; for the weight of the parts which compose it may be regarded as forces applied to this cord, and which must necessarily deflect it from a straight line.

Corollary IV.

Fig. 51.



133. If any number of powers P, Q, R, S, T, . . . act upon each other by cords joined together three by three in the same knot, it is easy, from what precedes, to find the ratios which these pow-

ers should have with each other as to their directions, so as to be in equilibrium. For the general equilibrium cannot take place, unless, 1st, the three powers joined together in each knot are in equilibrium with each other; 2d, each of the cords AB, BC, which join two knots, are equally stretched in the two directions. Hence, by naming V, X, the tensions of the two cords AB, BC, we will have (129), by reason of the equilibrium at the knot A,

$$\text{P} : \text{Q} :: \sin \text{QAB} : \sin \text{PAB},$$
$$\text{P} : \text{V} :: \sin \text{QAB} : \sin \text{PAQ};$$

by reason of the equilibrium at the knot B,

$$\text{V} : \text{R} :: \sin \text{RBC} : \sin \text{ABC},$$
$$\text{V} : \text{X} :: \sin \text{RBC} : \sin \text{ABR};$$

by reason of the equilibrium at the knot C,

$$\text{X} : \text{S} :: \sin \text{SCT} : \sin \text{BCT},$$
$$\text{X} : \text{T} :: \sin \text{SCT} : \sin \text{BCS}.$$

And by continuing these proportions, we will find the ratio of any two of these powers, and the ratio of one of them to the tension V, X of any cord whatever.

For example, by multiplying in order the 2d proportion and the 3d, we find

$$\text{P} : \text{R} :: \sin \text{QAB} \times \sin \text{RBC} : \sin \text{PAQ} \times \sin \text{ABC};$$

by multiplying the 2d and 4th,

$$P : X :: \sin QAB \times \sin RBC : \sin PAQ \times \sin ABR\,;$$

by multiplying the 2d, 4th and 5th,

$$P : S :: \sin QAB \times \sin RBC \times \sin SCT$$
$$: \sin PAQ \times \sin ABR \times \sin BCT\,;$$

by multiplying the 2d, 4th and 6th,

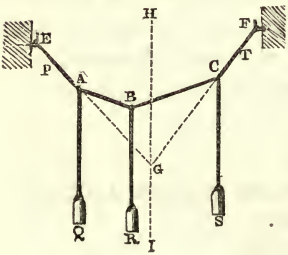
$$P : T :: \sin QAB \times \sin RBC \times \sin SCT$$
$$: \sin PAQ \times \sin ABR \times \sin BCS,$$

and so on.

It also follows from this, that the three cords united by the same knot are in the same plane (126), although those which are united at two knots may be in different planes.

Corollary V.

Fig. 52.



134. If the forces Q, R, S, be weights suspended by the knots A, B, C, to the same cord EABCF, and this cord be retained at its extremities by two fixed points E, F:

1st. The main cord and the cords of the weights Q, R, S, are in the same vertical plane;

for the two parts EA, AB of the cord are in the vertical plane drawn through the cord AQ; in like manner, the two parts AB, BC are in the vertical plane drawn through BR: now these two vertical planes pass through the same line AB, and coincide; hence the parts EA, AB, BC of the cord, and the directions of the cords AQ, BR, are in the same vertical plane. In the same manner it may be demonstrated that the part CF of the cord and the direction CS are in this same plane, and so on.

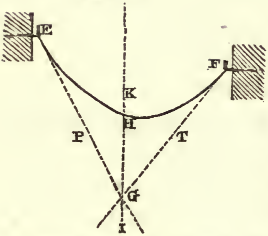
2d. The tensions of the two extreme parts of the cord are to each other reciprocally as the sines of the angles which these parts make with the vertical; for the angles QAB, ABR, are supplements of each other, and have the same sine: it is the same with the angles RBC, BCS, and so of the rest; hence, by neglecting the common factors in the proportion which gives the ratio of the two extreme tensions P, T (133), we have

$$\mathrm{P} : \mathrm{T} :: \sin \mathrm{SCT} : \sin \mathrm{PAQ}.$$

3d. The vertical HI, drawn through the point of intersection G of the prolongations of the two extreme parts of the cord, passes through the centre of gravity of the system of all the weights Q, R, S, . . .; for the two extreme parts being in the same plane, their tensions have a resultant whose direction passes through the point G; moreover, these tensions supporting the system of weights Q, R, S, . . . their resultant should be vertical, and pass through the centre of gravity of these weights; hence the point G and the centre of gravity of the weights Q, R, S, are in the same vertical.

Corollary VI.

Fig. 53.



135. When a heavy cord EHF is suspended in equilibrium to the two fixed points E, F, we may consider its axis as a thread without weight, charged with a weight distributed throughout its whole extent: hence 1st, this axis is in the vertical plane drawn through the two points of suspension; 2d, if the directions of the two extreme elements of this axis be prolonged to EG, FG, and through the point of intersection G we draw the vertical IH, the tensions of these two elements are to each other reciprocally as the sines of the angles which these elements make with the vertical: that is to say, by naming P and T these tensions, we have

$$P : T :: \sin IGF : \sin IGE;$$

3d, the centre of gravity K of the cord is in the vertical IH.

Finally, by considering the total weight Z of the cord as a force applied to the point G of its direction, we will find the efforts which the cord makes upon the two points of support E, F, along the directions EG, FG, by decomposing the force Z into two others which act in these directions; and we shall have (129)

$$Z : P : T :: \sin EGF : \sin IGF : \sin IGE.$$

Remark.

136. So far we have supposed that there were only three cords united at each knot, because, if the cords assembled at the same knot are greater in number and included in the same plane, it is not sufficient to know their directions in order to find what ratios the applied powers should have so as to be in equilibrium: that is to say, these ratios may vary in an infinite number of ways, without the forces ceasing to be in equilibrium.

Thus, whatever may be the number of powers directed in the same plane, it suffices, in order to be in equilibrium about the same knot, that the resultant of any two of them is equal and directly opposed to the resultant of all the others; hence all these forces, except two, being taken at pleasure, which determine the intensity and direction of the resultant, we can find the intensities of the latter two forces which will make an equilibrium with this resultant.

However, when four cords, joined at the same knot, are not in the same plane, their directions being given, the ratios of the intensities which the applied forces must have, to produce an equilibrium, are determined: for we have seen (44), that these forces must be to each other as the diagonal, and the adjacent sides of the parallelopipedon constructed on their lines of direction. But when the forces are not directed in the same plane, and their number is greater than four, the ratios of the forces are no longer determined by the knowledge of the direction of the cords.

Article II.

On the Equilibrium of the Lever.

Fig. 54.

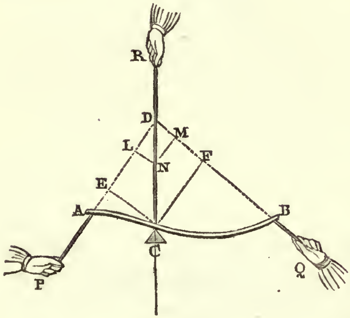
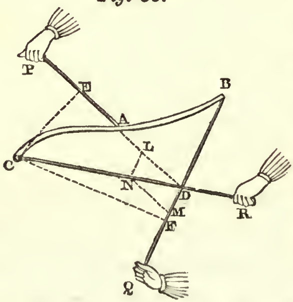


Fig. 55.



137. The lever is an inflexible rod ACB (*Fig.* 54), or CAB (*Fig.* 55), either straight or curved and moveable around one of its points C, rendered fixed by means of any obstacle; and this obstacle is termed the *point of support or fulcrum.*

138. First, supposing the lever to be without weight, and that it cannot in any manner slip upon its fulcrum, let P, Q be two powers applied, either immediately or by means of the cords AP, BQ, to the two points A, B of a lever. If we consider the resistance of the point C as the effect of a third force R applied to the lever at this point, we have seen, in order that equilibrium may subsist between these three forces, 1st, their directions must be included in the same plane, and meet in the point D (10); 2d, the forces P, Q must be to each other reciprocally as the perpendiculars CE,

CF, let fall from the fulcrum upon their directions (35): that is to say, we must have

$$P : Q :: CF : CE;$$

3d, if we lay off from the point D the lines DL, DM, upon the directions of the forces P, Q, proportional to the intensities of these forces, and finish the parallelogram DLMN; the diagonal DN will represent in intensity and direction the force R, and consequently the resistance of the fulcrum (35); thus we will have

$$P : Q : R :: DL : DM \text{ or } NL : DN,$$

or (129),

$$P : Q : R :: \sin QDR : \sin PDR : \sin PDQ.$$

Corollary I.

139. If we leave out of consideration the resistance of the fulcrum: that is to say, if we suppose this point to be capable of an indefinite resistance, it is necessary, in order that the two powers P, Q may be in equilibrium around this point by means of the lever, 1st, their directions and the fulcrum should be in the same plane; 2d, the two forces P, Q should tend to turn the lever around the fulcrum C in opposite directions, and their moments, referred to this point, should be equal: that is to say, we should have (80)

$$P \times CE = Q \times CF.$$

Corollary II.

140. Hence it appears, 1st, that however small the power Q may be, we may always, by means of a lever, put it in equilibrium around a fulcrum C, with another power P of given intensity and direction; for the direction of the force P being known, the distance CE of this direction from the fulcrum will be known, and we will know the moment $P \times CE$: hence it will be sufficient to arrange it so that the moment $Q \times CF$ of the power, is equal to the preceding: that is to say, to direct this power in such a manner that its distance CF from the fulcrum is equal to $\frac{P \times CE}{Q}$, and that it tends to turn the lever in the opposite direction to the force P.

2d. If the distance CF of the direction of the force Q from the fulcrum is known, we will find the intensity which this force must have in order to produce an equilibrium with the force P, by dividing the moment of this latter force by the distance CF: that is to say, we shall have

$$Q = \frac{P \times CE}{CF}.$$

Corollary III.

141. The effort or load, which the fulcrum C sustains, being equal to the resultant of the two forces P, Q, we will find this load by means of the following proportion:

$$P : Q : R :: \sin QDR : \sin PDR : \sin PDQ :$$

that is to say, we shall have

$$R = \frac{P \times \sin PDQ}{\sin QDR},$$

or,

$$R = \frac{Q \times \sin PDQ}{\sin PDR}.$$

Corollary IV.

142. Hence, if the fulcrum does not possess indefinite resistance, in order that it may not be moved, and that equilibrium may subsist, it is necessary that the resistance in the direction CD, should be equal to the resultant of the two forces P, Q: that is to say, equal to

$$\frac{P \times \sin PDQ}{\sin QDR};$$

or, which is the same, to $\dfrac{Q \times \sin PDQ}{\sin PDR}$.

Corollary V.

143. In general, of the six things to be considered in the equilibrium of the lever, namely the intensities and directions of the two powers P, Q, and those of the load on the fulcrum, any three being given, the other three may be determined in all cases; or of the six

analogous things which may be considered in the triangle DLN, namely, the sides and the angles, three being given, we can determine the others.

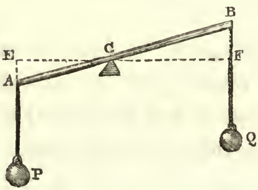
Remark I.

144. If the lever can slip upon the fulcrum, the conditions, which have just been given, are not sufficient to keep the lever stable, and to produce an equilibrium; it is necessary, besides, that the direction DC of the load on the fulcrum should be perpendicular to the surface of the lever at the point C; for, if this direction were oblique, the lever would have a tendency to slip towards the side of the greater angle, and in fact would slip, whenever this tendency should be greater than the friction upon the fulcrum which opposes this effect, as we will demonstrate in treating of the inclined plane.

Remark II.

145. What has just been stated contains the whole theory of the equilibrium of two powers applied to a lever, considered to be without weight, and retained by a fulcrum; we will now make application of it to a few simple cases.

Fig. 56.



If the directions of the powers P, Q, applied to a lever, are parallel to each other: for example, if there are two weights suspended at the points A, B, the load, which the fulcrum C sustains, is equal to their sum

$P+Q$, and the two perpendiculars CE, CF, let fall from the fulcrum upon their directions, are in a straight line. Hence, if the lever be straight, the triangles ACE, BCF will be similar, and give

$$CF : CE :: CB : CA.$$

Hence we shall have, in the case of equilibrium,

$$P : Q :: CB : CA :$$

that is to say, the weights P, Q will be to each other reciprocally as the arms of the lever.

Thus, the intensity and the lever arm of a resistance P being given, 1st, the lever arm, which must be given to a power Q in order to produce an equilibrium with it, will be

$$CB = \frac{P \times CA}{Q}.$$

2d. The intensity of the power necessary to be applied to the given point B, in order to produce an equilibrium with it, will be

$$Q = \frac{P \times CA}{CB}.$$

Finally, if the two weights P, Q, and the length AB of the lever, be given, we will find the fulcrum C, around which these weights will be in equilibrium, by dividing the lever AB into two parts reciprocally proportional to the two weights.

Remark III.

146. When there are more than two powers applied to the same lever, it is not sufficient to know their directions, and the position of the fulcrum, in order to determine the ratios which they should have to each other so as to be in equilibrium; but, as equilibrium cannot take place between several forces about a fulcrum, unless the resultants of all the forces are destroyed by the resistance of this point, it is evident, in this case, that the conditions of equilibrium are reduced to the two following: 1st, all the forces must have a single resultant; 2d, the direction of this resultant must pass through the fulcrum.

Corollary.

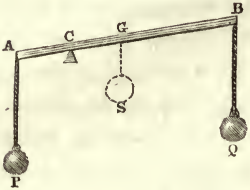
147. If the directions of all the forces are included in the same plane, these forces have necessarily a single resultant (43), and the first condition is fulfilled; hence it suffices for the equilibrium, that the direction of this resultant passes through the fulcrum, or, which amounts to the same, the sum of the moments of the forces, which tend to turn the lever in one direction around the fulcrum, is equal to the sum of the moments of those which tend to turn it in the opposite direction.

Remark IV.

148. So far we have abstracted weight from the lever; but if this weight enter into consideration, it must be regarded as a new force, applied to the centre

of gravity of the lever in a vertical direction; and in the case of equilibrium, the conditions just stated subsist between all the forces, including that under consideration.

Fig. 57.



Let P, Q be two weights suspended from a heavy lever AB, and in equilibrium about the fulcrum C: we will consider the weight of the lever as a third weight S, suspended from the centre of gravity of the lever, and the sum of the moments of the two weights Q, S, referred to the fulcrum C, will be equal to the moment of the weight P: that is to say, we shall have

$$\text{Q}\times\text{CB}+\text{S}\times\text{CG}=\text{P}\times\text{CA};$$

or, subtracting from these two equal quantities the moment of the lever $\text{S}\times\text{CG}$,

$$\text{Q}\times\text{CB}=\text{P}\times\text{CA}-\text{S}\times\text{CG}.$$

Thus, knowing the length and the weight of the lever, the position of its centre of gravity, that of the fulcrum and one of the two weights P, Q, it will always be possible to get the other weight; for we shall have

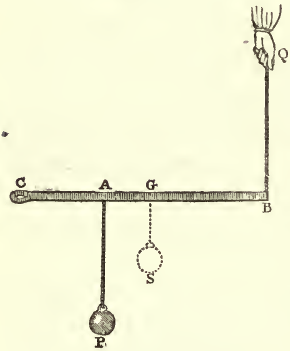
$$\text{P}=\frac{\text{Q}\times\text{CB}+\text{S}\times\text{CG}}{\text{CA}},$$

and

$$\text{Q}=\frac{\text{P}\times\text{CA}-\text{S}\times\text{CG}}{\text{CB}}.$$

The load at the fulcrum is evidently equal to the sum of the weights P+Q+S.

Fig. 58.



149. But, if the weight P, suspended from the heavy lever AB, be retained in equilibrium about the fulcrum C, by means of a vertical power Q, and directed upwards, the moment of the force Q, which tends to turn the lever in one direction, will be equal to the sum of the moments of the weights P, S, which tend to turn it in the opposite direction, and we shall have

$$Q\times CB = P\times CA + S\times CG,$$

or, subtracting the moment of the lever,

$$Q\times CB - S\times CG = P\times CA.$$

Thus the intensities, which the two forces P, Q must have in order to be in equilibrium, will be

$$P=\frac{Q\times CB - S\times CG}{CA},$$

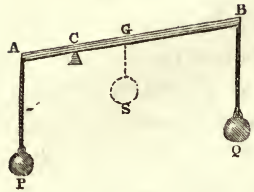
and

$$Q=\frac{P\times CA + S\times CG}{CB},$$

and the weight upon the fulcrum P+S−Q.

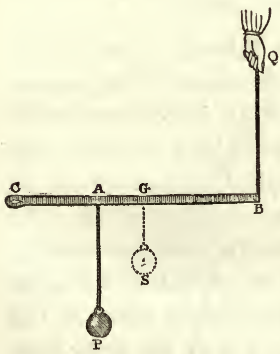
COROLLARY.

Fig. 57.



150. Hence it appears by regarding the weight P as a resistance, and the force Q as a power which has to bring the weight into equilibrium or to overcome it, by means of the lever AB, the weight of this lever is a force which may either increase or diminish the power, according as this weight tends to turn the lever in the same direction as the power, or in the opposite direction. For example, in the case of *Fig.* 57, the weight of the lever increases the power Q; and by prolonging the lever arm CB of this power, it would be placed in the condition of producing an equilibrium with a greater resistance, for two reasons: 1st, because its moments would be thus augmented; 2d, because the weight S of the lever would be augmented, which alone would produce an equilibrium with a greater part of the resistance. But, in the case of *Fig.* 58, the weight of the lever diminishes the power Q, and we cannot increase the length of the lever arm CB without at the same time augmenting its weight S, which forms part of the resistance; thus, in order that there may be in this case an

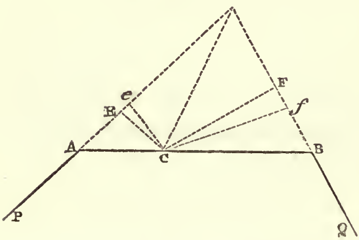
Fig. 58.



advantage in prolonging the arm of the lever, it is necessary that the moment of this prolongation should be less than the resulting increase in the moment of the power.

THEOREM.

Fig. 59.



151. *Two powers* P, Q, *applied to a lever* AB, *and in equilibrium about a fulcrum* C, *are to each other reciprocally as the spaces which these powers traverse in the line of their directions, if the equilibrium be disturbed infinitesimally.*

Demonstration. From the fulcrum C let fall the perpendiculars CE, CF upon the directions of the powers; and instead of the straight lever AB, let us consider the bent lever ECF; at the extremities E, F of which conceive the powers P, Q to be applied; then suppose, by virtue of a derangement in the equilibrium, that the bent lever ECF takes the infinitely near position eCf. This being done, the small arcs Ee, Ff, will be the spaces which the powers P, Q would traverse by virtue of this derangement. Now, the angle ECF of the bent lever being invariable, the two angles ECe, FCf are equal, and we have

$$CF : CE :: Ff : Ee;$$

moreover, because of the equilibrium, we have (35)

$$P : Q :: CF : CE;$$

hence we have also

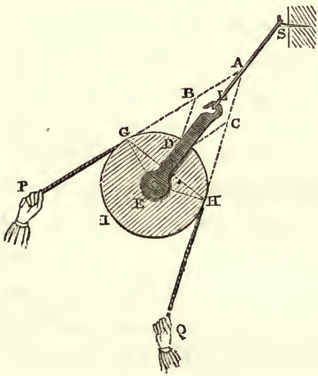
$$P : Q :: Ff : Ee.$$

We will have occasion to show, in a subsequent part, that an analogous proposition occurs in cases of equilibrium for all other machines.

On Pulleys.

I.

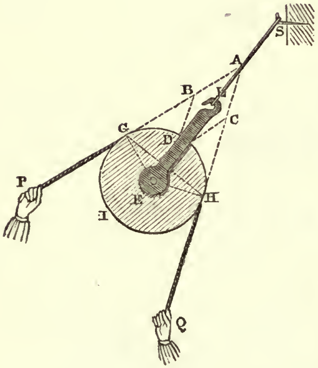
Fig. 60.



152. A *pulley* is a wheel having a groove on its circumference to receive a cord PGDHQ, and is traversed at its centre by an axle E, upon which it turns in a sheath or block EL.

II.

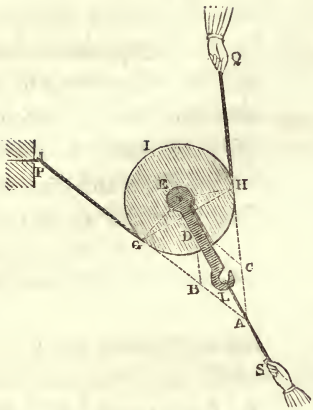
Fig. 60.



153. Suppose the axis of the pulley being fixed, two forces P, Q are applied to the extremities of the cord, and this cord being perfectly flexible exerts no friction upon the groove of the pulley, so that it may slide freely on this rim. Whatever the figure of the pulley may be in other respects, that is to say, whether the arc GDH, embraced by the cord, be circular or not, it is evident that the two forces cannot be reciprocally in equilibrium unless they are equal; for if they were unequal, the greater would overcome the smaller by causing it to slide in the groove of the pulley.

Upon the same supposition, the pulley, having no other fixed point than its centre, and being drawn by the two forces P, Q, cannot remain at rest, unless the resultant of these two forces is directed towards the centre, and is destroyed by the resistance of this point. Hence, having prolonged the directions PG, QH of the two cords, until they meet in some point A, and taking upon these directions the equal lines AB, AC to represent the forces P, Q, if the parallelogram ABDC be completed, the diagonal AD, which will represent the resultant of these two forces, must pass through the fixed point E.

Fig. 61.



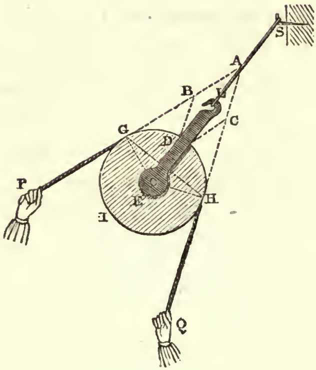
Now, when the figure is circular, this last condition is always fulfilled: thus, the triangle ABD being isosceles, the angle BAD is equal to the angle BDA, and consequently to the angle DAC; hence the diagonal AD divides the angle BAC into two equal parts. But, if we draw the line EA, this line divides the same angle into two equal parts; for if, through the centre E, we draw, to the points of contact of the cords, the radii EG, EH, perpendicular to the directions of these cords, the two rectangular triangles EGA, EHA will be in all respects equal, and we shall have the angle EAG of the one, equal to the angle EAH of the other. Hence the line EA, and the diagonal DA, will have the same direction.

Hence, the centre of a pulley being fixed and its figure circular, it is necessary that the two forces P, Q, to be in equilibrium, should be equal, and that the pulley remain at rest, at the same time, about its axis.

The load which the axis of the pulley sustains is evidently equal to the resultant of the two forces P, Q; hence, if we name R this load, we shall have (36)

$$P : Q : R :: AB : AC : AD.$$

Fig. 60.



Finally, draw the cord GH of the arc embraced by the rope; the two triangles GHE, ABD will be similar, because they will have their sides perpendicular each to each; and we shall have

AB : AC : AD : : GE : EH : GH;

hence we shall have

P : Q : R : : GE : EH : GH.

III.

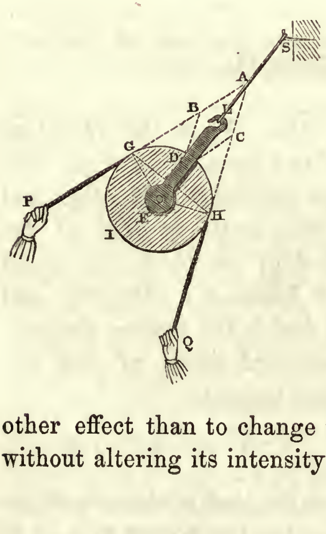
154. If the axis of the pulley is not absolutely fixed, but retained simply by the power S, by means of the block EL and the cord LS; in order that this axis may be at rest, and the three forces P, Q, S in equilibrium, it is necessary that the force S should be equal and directly opposed to the load which the axis sustains. Hence, 1st, the direction of this force should coincide with the line EA; 2d, its intensity should be equal to the resultant R of the two forces P, Q, and we shall have

P : Q : S : : GE : EH : GH.

Thus, *when two forces* P, Q, *applied to a rope embracing a pulley, are in equilibrium with each other, and with a third force* S *applied to the axis of the pulley,* 1st, *the two forces* P, Q *are equal to each other;* 2d, *the direction of the force* S *bisects the angle formed by the directions of the two others;* 3d, *each of the two forces* P *and* Q *is to the third force* S, *as the radius of the pulley is to the chord of the arc embraced by the rope.*

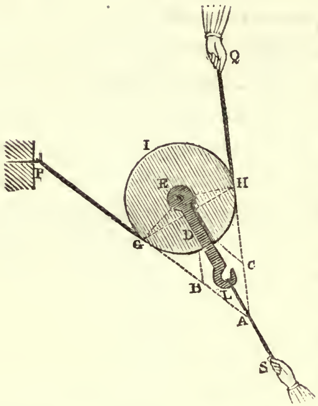
Corollary I.

Fig. 60.



155. We observe that if the cord, fastened to the block, instead of being drawn by a force S, is attached to a fixed point of indefinite resistance, and if it be proposed, by employing the pulley, only to place the two forces P, Q in equilibrium, or to overcome a resistance P, by means of a power Q, the pulley does not assist the power: it has no other effect than to change the direction of this force, without altering its intensity.

Fig. 61.



But if one of the extremities of the cord, which embraces the pulley, is attached to a fixed point P, and, by using the pulley, we design to place a power Q in equilibrium with a resistance S, attached to the block, the pulley assists the power, which is always less than the resistance; for we have

Q : S :: EH : GH.

Corollary II.

Fig. 62.

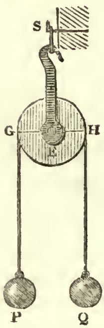
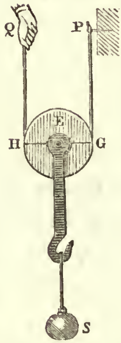


Fig. 63.



156. When the directions of two parts PG, QH of the cord are parallel to each other, and consequently to that of the cord of the block, the chord GH becomes a diameter, and is double the radius; the proportional series of No. 153 then becomes

P : Q : S :: 1 : 1 : 2,

that is to say, the force S, or the load of the axis of the pulley, is equal to the sum of the two powers P, Q, or to

double one of them. Thus, in the case of *Fig.* 63, the power Q, which, by means of the pulley and the point of support P, will produce an equilibrium with the resistance S, will be only half of this resistance.

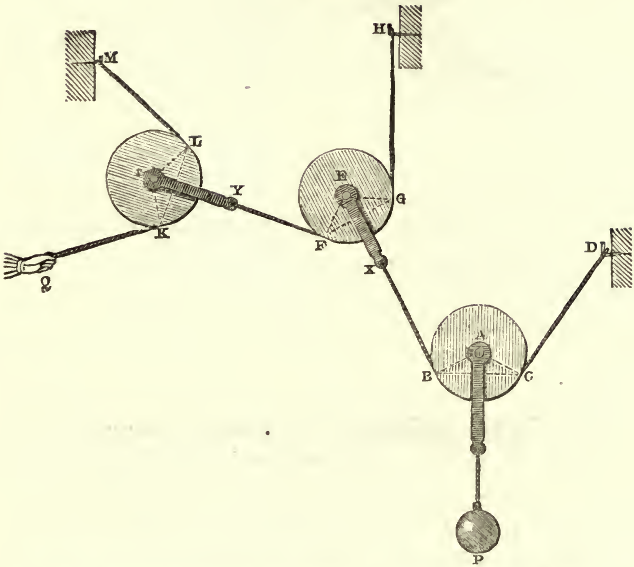
IV.

157. A pulley is said to be *immoveable* when its block is attached to a fixed point and the power and resistance are applied to the cord which embraces it; but when the resistance is attached to the block, and the pulley has to move with it, as in *Figs.* 61, 63, the pulley is said to be *moveable.*

This being established, let there be any number of moveable pulleys, (*Fig.* 64,) and considered without weight; let the first bear a weight P suspended to its hook, and embraced by a cord, one of whose extremities is attached to the fixed point D, and the other is applied to the block of the second pulley; let this pulley be embraced by another cord, one of whose extremities is fixed to the point H, and the other attached to the block of the next pulley; let this third pulley be embraced by a third cord fixed on one side to M, and drawn on the other by a power Q; and so on, if the number of pulleys be greater. Finally, supposing the whole system to be in equilibrium, draw the radii and the subtending chords of the pulleys as is shown in the figure. We may consider the equilibrium of the first pulley A as though this pulley were alone; and representing by X the tension of the cord BX, we shall have (155)

$$P : X :: BC : BA.$$

Fig. 64.



For the same reason, calling Y the tension of the cord FY, we shall have

X : Y : : FG : EF.

We shall have, in like manner, for the third pulley,

Y : Q : : KL : IK ;

and so on, whatever may be the number of pulleys. Hence, by multiplying in order all these proportions, we shall have

P : Q : : BC×FG×KL : BA×EF×IK :

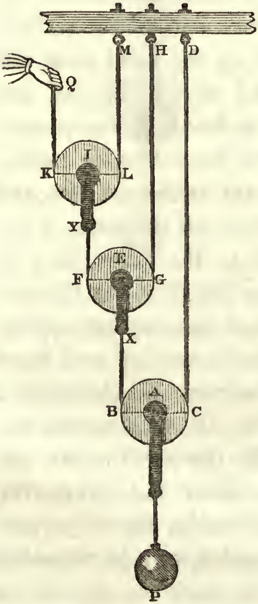
that is to say, the resistance is to the power as the product of the subtending chords is to the product of the radii.

COROLLARY.

158. When all the ropes CD, GH, LM, . . are parallel, the subtending chords will be diameters, and the preceding proportion will become

$$P : Q :: 2\times2\times2 : 1\times1\times1, \text{ or } :: 8 : 1;$$

Fig. 65.



from which it is evident, that, in this case, the resistance is to the power as the number 2, raised to a power indicated by the number of moveable pulleys, is to unity.

Thus, by suitably increasing the number of moveable pulleys, we may put a moderate force in equilibrium with a very great resistance. For example, with three pulleys, and by means of the fixed points D, H, M, the power produces an equilibrium with a resistance eight times greater than itself.

However advantageous this disposition of moveable pulleys may at first appear, it is rarely

employed; because, in order to make the first pulley A traverse a certain space, it is necessary that the second should traverse a double space, and the third a quadruple; and so on, which requires too much room; and thus, most generally, muffles are used instead.

V.

159. The term *muffle* is applied to a system of several pulleys assembled in the same block and upon separate axes, as in *Figs.* 66, 67, or upon the same axis, as in *Fig.* 68. A fixed muffle and a moveable muffle are always employed at the same time; and all the pulleys of the two muffles are embraced by the same cord, one of whose extremities is attached to one of the two muffles, and the other extremity is drawn by the power; and the resistance is suspended to the moveable muffle.

We may give different diameters to the pulleys, and arrange them in such a manner that all the parts of the cord, which go from one muffle to the other, may be parallel to each other, as in *Figs.* 66, 67; but this arrangement increases the extent of the muffles. They may be reduced to a volume much smaller and more convenient, by mounting in each of them all the pulleys upon the same axis, as in *Fig.* 68. By this means, the cords which are on one side of the muffles are not parallel to those on the other side; but, when the distance of the muffles is considerable, the departure from parallelism is very small, and it may be regarded as insensible.

Fig. 66.

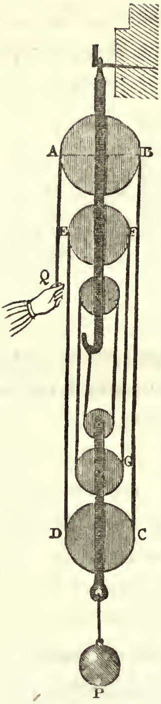
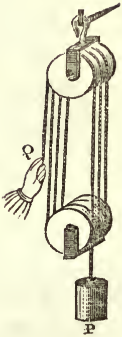


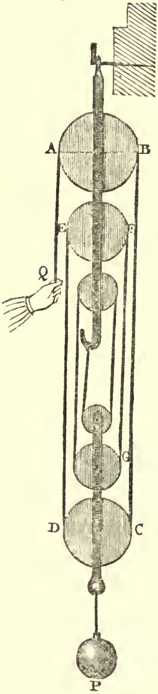
Fig. 67.

Fig. 68.



160. By considering, then, the cords of the muffles as parallel to each other, and abstracting all weight from the whole machine, let Q be a power in equilibrium with the resistance P, suspended from the block of the moveable muffle. Equilibrium cannot exist throughout the whole machine, unless it occur in each particular pulley, and the two parts of the cord which embraces this pulley are equally stretched (154). Thus, the tensions of the two cords QA, BC are equal to each other; so likewise are those of the two cords BC, DE, and those

Fig. 66.



of the cords DE, FG, and so on, whatever may be the number of the cords; hence all the cords which go from one muffle to the other are equally stretched. Now the sum of these tensions produces an equilibrium with the resistance P, and is equal to it; or, what is the same, the tension of one of the cords, multiplied by their number, is equal to the resistance; hence the tension of one of the cords, or the power Q, is the quotient of the resistance P divided by the number of cords which go from one muffle to the other.

Hence it is evident, that, in the case of *Fig.* 66, where the extremity of the cord is attached to the fixed muffle, the power Q should be one-sixth of the resistance to be in equilibrium with it, and in the case of *Fig.* 67, where the extremity of the cord is attached to the moveable muffle, it should be one-fifth, because there is one pulley and one cord less.

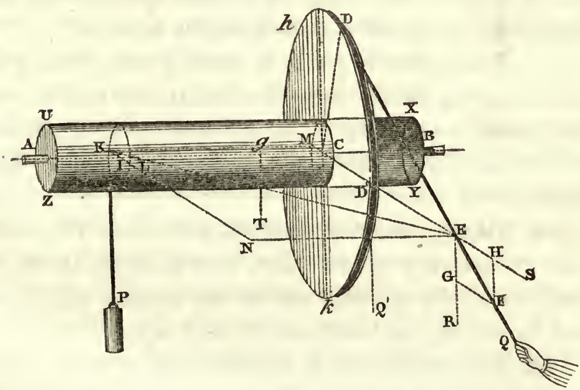
If it be desired to introduce the consideration of the weight of the moveable muffle, it may be regarded as constituting part of the resistance.

Of the Wheel and Axle.

I.

161. The *wheel and axle*, *windlass*, or *capstan*, is a machine consisting of a cylinder moveable upon an axis, and of a cord, which, having one of its extremities wound around the cylinder, while a power Q causes it to turn, overcomes a resistance P attached to its other extremity. The cylinder is furnished at its two ends with trunnions A, B, bearing upon supports, which are named *boxes*, and by means of which it turns freely on its axis.

Fig. 69.



162. There are several modes of applying the power to this machine, to communicate the movement of rotation to the cylinder.

1st. We may unite solidly with the cylinder and upon the same axis, a wheel, whose circumference, having a groove in it like that of a pulley, is surrounded by a second cord. This cord, being drawn by the power, causes both the wheel and the cylinder to turn upon their common axis. This first arrangement, to which we will refer all the others, is rarely employed, because it requires the cord of the wheel to be very long, while the space to be traversed by the resistance is inconsiderable.

2d. The rim of a wheel, furnished with spokes placed equally far apart, and to which men's hands are applied, furnishes them with the means of making use of part of their weight in turning the machine.

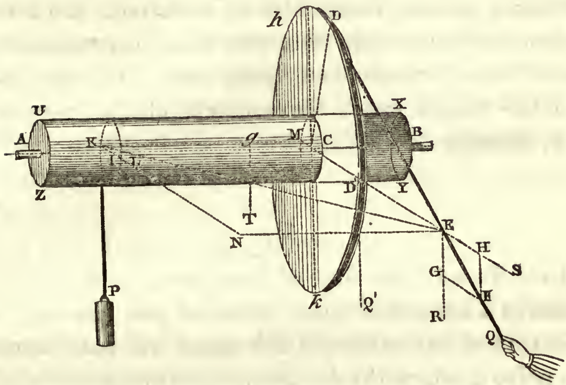
3d. In other cases, instead of the wheel, there is mounted upon the cylinder a large hollow drum in which men or animals can walk; and then by their feet they cause both the drum and the cylinder to turn.

4th. Sometimes, in place of making use of the wheel and the drum, the cylinder is traversed by bars perpendicular to its axis, and at the extremities of which men act by the force of their muscles and part of their weight.

5th. Finally, most frequently, and when the resistance is not very great, there are adapted to the extremities of the cylinder one or two cranks, which men turn by employing the force of their arms.

163. The names of this machine vary with its object, and even with its position. Ordinarily it is named *windlass*, and *wheel and axle*, when the cylinder is horizontal; and *capstan*, when the cylinder is vertical, and horizontal bars are used to put it in motion.

Fig. 69.



It is evident that the different modes in which the power may be applied to the cylinder of the wheel and axle, can all be referred in theory to the first we have described; for, whatever may be the direction of the power, when it is directed in a plane perpendicular to the axis of the cylinder, we may always conceive it to be applied to a wheel whose circumference would be tangent to the direction of this power. Thus we will suppose that VXYZ being the cylinder of the machine, whose axis ACB is perpendicular to the plane of the wheel *h*DD′*k*, 1st, the power Q is applied to the circumference of this wheel in any direction DQ, contained in the plane of the wheel, and tangent to the circumference of the radius CD, at a given point D; 2d, that the direction KP of the resistance is tangent at K to the surface of the cylinder, and situated in a plane parallel to that of the wheel. Lastly, to render the conception clear, we will suppose that the axis AB of the cylinder is horizontal, and consequently that the direction KP of

the resistance is vertical. This being granted, two questions present themselves to be solved: the first is to find the ratio which the power Q and the resistance P should have to produce an equilibrium; the second is to find the weight which the supports of the two pivots A, B, sustain.

II.

164. To solve the first of these two questions, let us conceive a horizontal plane KMEN to pass through the axis AICB of the cylinder: this plane will pass through the point K, where the direction of the resistance touches the surface of the cylinder; moreover, it will intersect the plane of the wheel in a horizontal line ME, which will pass through the centre C, and it will meet the direction of the power Q in some point E. Prolong the line ME to ES, and through the point E draw the vertical ER; the three lines EQ, ER, ES being included in the same plane, which is that of the wheel, we can decompose the power Q into two forces R, S, directed along EG, EH. For this purpose, we will represent this power by the parts EF of its direction; and completing the parallelogram EGFH, we will have

$$Q : S :: EF : EH,$$

$$Q : R :: EF : EG, \text{ or } FH;$$

or, since, by drawing the radius CD, the two rectangular triangles CDE, EHF will be similar, and give

$$EF : FH : EH :: CE : CD : DE,$$

we shall have

$$Q : S :: CE : DE,$$
$$Q : R :: CE : CD.$$

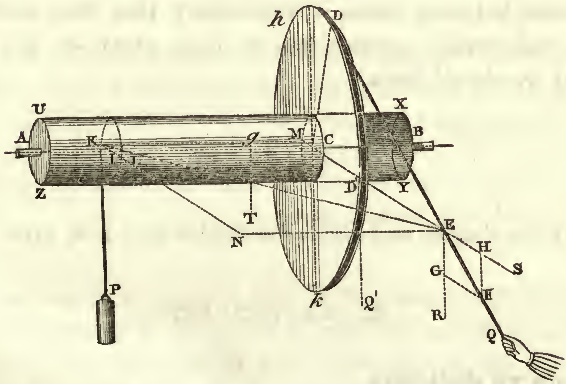
Thus, in place of the power Q, we can take the two forces R, S, whose values

$$R = \frac{Q \times CD}{CE},$$

$$S = \frac{Q \times DE}{CD},$$

are known, since the direction of the force Q being given, all parts of the triangle CDE are known.

Fig. 69.



Now, of the two forces R, S, the latter being directed towards the axis of the cylinder which is immoveable, it may be regarded as immediately applied to the point

C, and destroyed by the resistance of the axis; this force, then, cannot contribute in any manner to the motion of rotation of the cylinder, and it has no other effect than that of pressing the pivots against their supports. Hence, there remains only the force R which can be employed to produce an equilibrium with the resistance P.

Now, in the horizontal plane KMEN draw the radius of the cylinder KI, which will be perpendicular to the axis and parallel to ME; also draw the line KE which will intersect the axis in some point L. This being done, the point L, being in the axis, may be regarded as immoveable, and the line KE may be taken for an inflexible rod, retained by a fulcrum L, at the extremities of which are applied the two forces R, P. Now the directions of these two forces are both vertical, and consequently parallel; hence in order that equilibrium may subsist between them, it is necessary that they should be reciprocally proportional to their arms LE, LK, or that we should have

$$R : P :: KL : LE.$$

But the similar rectangular triangles KIL, LCE, give

$$KL : LE :: KI : CE;$$

hence we shall have

$$R : P :: KI : CE.$$

Hence by multiplying in order, this proportion and the following

$$Q : R :: CE : CD,$$

which we found above, we shall have, in case of equilibrium,

$$Q : P :: KI : CD;$$

that is to say, *the power will be to the resistance as the radius of the cylinder is to the radius of the wheel;* which is the answer to the first question.

By making the product of the extremes equal to that of the means in the above proportion, we have

$$Q \times CD = P \times KI.$$

The line CE meets the circumference of the wheel at the point D'; imagining that there is through this point a force Q' equal to Q directed along the tangent $D'Q'$, we shall have (31)

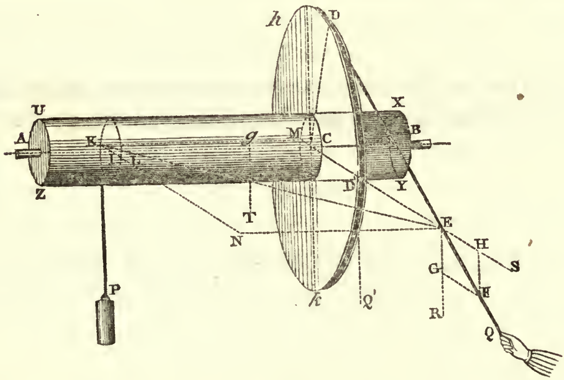
$$Q \times CD = Q' \times CD' = P \times KI;$$

from which it appears, that in the case of equilibrium, the moments of the power Q', equal to Q, and of the resistance P, both taken with reference to the vertical plane passing through the axis of the cylinder, are equal to each other.

III.

165. The pressure which the trunnions exert against their points of support is evidently the effect only of the forces P, Q, and of the weight T of the machine, which may be considered as united at its centre of gravity g, where, by taking the two forces R, S, instead of the power Q, these pressures are produced by the four forces P, R, S, T.

Fig. 69.



These forces are all known; for, 1st, the resistance P and the weight T of the machine are given immediately; 2d, the two other forces R and S, whose values we have found to be, in general,

$$R = \frac{Q \times CD}{CE},$$

$$S = \frac{Q \times DE}{CD},$$

become, in the case of equilibrium where we have $Q \times CD = P \times KI$,

$$R = \frac{P \times KI}{CE},$$

$$S = \frac{P \times KI \times DE}{CD \times CE},$$

and containing only known quantities, since the direction of the power Q being given, we can find all the parts of the right-angled triangle CDE.

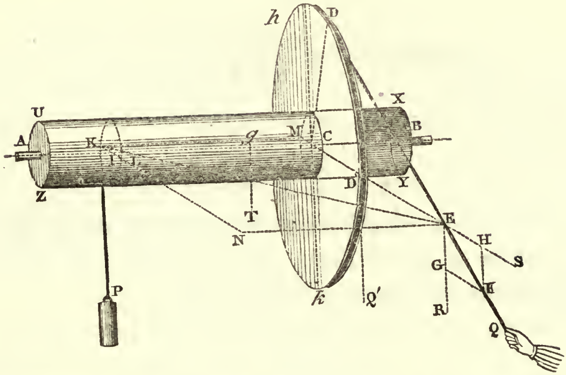
Now the two forces P, R, whose directions are vertical, and which are in equilibrium about the point L, exert upon this point of the axis a weight whose direction is vertical, and which is equal to their sum $P+R$. Moreover, this weight $P+R$, being sustained by the two points of support at A and B, must be regarded as the resultant of the two vertical pressures which it exerts at these points; and we will find each of these pressures, by dividing the resultant $P+R$ into two parts reciprocally proportional to the distances of the point L from the two supports. Let X be the pressure exerted at the point A, and X′ that exerted at the point B; we will find these two pressures by the following proportions:

$$AB : BL :: P+R : X,$$

$$AB : AL :: P+R : X'.$$

In like manner the weight T of the whole machine, supposed to be united at the centre of gravity g, may be regarded as the resultant of the vertical pressures

Fig. 69.



which it produces upon the two points of support: and we will find these pressures by dividing the weight T into two parts reciprocally proportional to the distances Ag, gB.

Then let Y and Y′ be the pressures which result respectively at the points A and B; we will find these pressures by the two proportions which follow:

$$\text{AB} : \text{B}g :: \text{T} : \text{Y},$$
$$\text{AB} : \text{A}g :: \text{T} : \text{Y}'.$$

Lastly, the horizontal force S, applied to the point C of the axis, produces horizontal pressures upon the two supports, directed perpendicularly to the axis AB, and of which it is the resultant: we will find likewise these pressures by dividing the force S into parts reciprocally proportional to the lines AC, CB. Then let Z, Z′ be the horizontal pressures, produced respectively upon the

points A, B: we will find these pressures by the proportions:

$$AB : BC :: S : Z,$$

$$AB : AC :: S : Z'.$$

Thus the point of support A sustains the two vertical pressures X, Y, and the horizontal pressure which acts in the direction of the force S. In like manner the point B sustains the vertical pressures X′, Y′, and the horizontal pressure Z′. Hence, by compounding, for each of these points, the forces which act upon it, we will find the intensity and direction of their resultant; and we shall have the intensity of the resistance of which it must be capable, as well as the direction in which it must resist, so as not to yield to the united efforts of the resistance P, the power Q, and the weight T of the machine; which is the object of the second question.

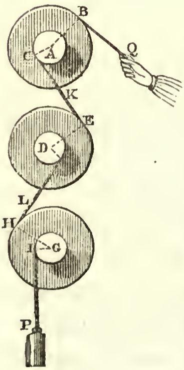
IV.

166. Hitherto we have regarded the cords as infinitely fine threads; but, when the weight P is suspended by the cord KP, the line of direction of this weight does not coincide with the axis of the cord; and in the case where the cord by wrapping around the cylinder does not change in figure, its axis is always at a distance from the surface of the cylinder equal to the semi-diameter of the cord. Thus, by reason of its thickness, the cord is in the same condition as though, being infinitely fine and reduced to its axis, it were wrapped

upon a cylinder whose radius was greater than the radius of the cylinder of the machine, by a quantity equal to the semi-diameter of the cord. It is the same with the cord of the wheel, which, by reason of its thickness, may be regarded as a mathematical line wrapped upon a wheel whose radius is greater than that of the wheel of the machine, by a quantity equal to the semi-diameter of this cord. Hence, in all the relations we have just found, it is necessary to augment the radius of the cylinder and that of the wheel by quantities respectively equal to the semi-diameters of the cords which envelope them. Thus, for example, in the case of equilibrium of the wheel and axle, *the power* Q *is to the resistance* P *as the radius of the cylinder, augmented by the radius of the cord* KP, *is to the radius of the wheel, augmented by the radius of the cord* DQ.

V.

Fig. 70.



167. If several wheels and axles are so arranged that the cord BQ of the first wheel, being drawn by a power Q, the cord CE of the cylinder of this wheel, instead of being attached immediately to the resistance, is wound around the second wheel; the cord FH of the cylinder of the second is likewise wound around the wheel of the third; and lastly, the cord IP of the cylinder of the last is applied to the resistance P; the power and the

resistance will not be in equilibrium, unless there is also equilibrium between the two forces which act upon each separate wheel. Thus, by naming K the tension of the cord CE, and L that of the cord FH, in the case of general equilibrium we have: 1st, by reason of the equilibrium of the first wheel and axle (164),

$$\text{Q} : \text{K} :: \text{CA} : \text{AB};$$

2d, by reason of the equilibrium of the second,

$$\text{K} : \text{L} :: \text{DF} : \text{DE};$$

3d, by reason of the equilibrium of the third,

$$\text{L} : \text{P} :: \text{GI} : \text{GH};$$

and so on, whatever may be the number of machines. Hence, by multiplying all these proportions in order, we have

$$\text{Q} : \text{P} :: \text{CA}\times\text{DF}\times\text{GI} : \text{AB}\times\text{DE}\times\text{GH};$$

that is to say, the power is to the resistance as the product of the radii of the cylinders is to the product of the radii of the wheels.

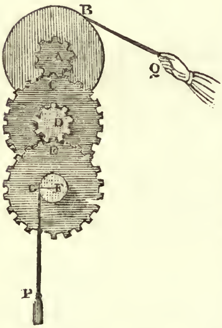
For example, if the radius of the wheel of each is four times the radius of its cylinder, in the case of equilibrium of all three wheels and axles, we have

$$\text{Q} : \text{P} :: 1\times1\times1 : 4\times4\times4, \text{ or } :: 1 : 64.$$

Hence we see, that by multiplying in this manner the number of machines we can put moderate powers in a state of equilibrium with very great resistances; but this arrangement is almost never employed, because it requires too great a length of cord in the first wheels, while the space which the resistance has to traverse is inconsiderable.

VI.

Fig. 74.



168. When we wish to profit by the advantages of this arrangement, 1st, the cords are suppressed which transmit the motion from one wheel and axle to another; 2d, on the circumference of each wheel, teeth are placed equally distant from each other; 3d, on the arbor of each of the *toothed-wheels*, another wheel similarly toothed, of a smaller diameter, and which is called the *pinion*, is fixed solidly; 4th, lastly, the whole system is so arranged that the teeth of each pinion interlock with the teeth of the following wheel. By this means one wheel cannot turn upon its axis, without its pinion communicating motion to the wheel with which its teeth are engaged, and causing it to turn upon its axis; and the number of revolutions which each pinion can cause the following wheel to make, is unlimited. The system is then in the same condition as though the pinion, considered as the cylinder of a wheel

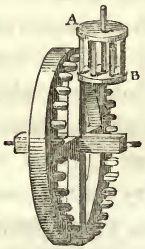
and axle, and the wheel with which it interlocks, were embraced by the same cord as in the preceding case.

Hence, when a power Q, applied to the circumference of the first wheel, is in equilibrium with a resistance P, applied to the circumference of the last pinion, the power is to the resistance as the product of the radii of the pinions is to the product of the radii of the wheels.

VII.

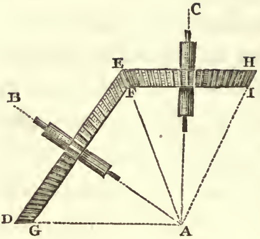
169. Toothed wheels are employed in a very great number of machines, principally in mills and in the works of time pieces. Their immediate object is to communicate to a cylinder or *arbor* a movement of rotation about its axis, by the aid of the rotary motion of another arbor. For this purpose it is not necessary that the axes of the two arbors should be parallel, as we have heretofore supposed; it is sufficient that they are in the same plane.

Fig. 72.



170. When the axes of the two arbors are at right angles, the teeth ordinarily are placed perpendicularly to the plane of the wheel, as may be seen in *Fig.* 72; then they can interlock with those of the pinion, or with the *staves* of the *trundle* AB, which produces the effect of a pinion: by this interlocking, the teeth of the wheel are forced to slide upon the staves in the direction of the axis of the trundle, which also causes friction.

Fig. 73.

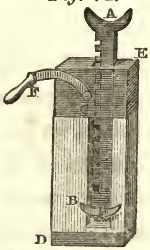


171. In general, whatever may be the angle BAC, which the axes of the two arbors make, so that the movement of rotation of the one is communicated to the other by means of two toothed wheels DEFG, EHIF, and the teeth do not slide upon each other in the direction of the axes, it is necessary that these two wheels, called *beveled wheels*, should be truncated sections of two cones DAE, EAH, which have the same summit A, and whose axes coincide with those of the arbors; moreover, the teeth of the two wheels should be terminated by conic surfaces which have for their summit the common point A.

VIII.

172. The *jack-screw* is also a machine which may be referred to the wheel and axle.

Fig. 74.



The simple jack-screw is composed of a bar of iron AB, furnished with teeth upon one of its sides, and moveable in the direction of its length within a box DE. The teeth of the bar interlock with those of a pinion C, which is turned upon an axis by means of a crank F. The teeth of the pinion carry along those of the bar, and cause the weight to rise, which rests upon the head A of the bar, or which is raised by the hook B. This machine, evidently, is nothing else than a wheel and axle, and, in the case of equilibrium, by

supposing the direction of the power to be perpendicular to the arm of the crank, the power is to the resistance as the radius of the pinion is to the arm of the crank.

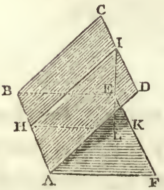
In the compound jack-screw, the teeth of the first pinion interlock with those of a second toothed wheel, and the teeth of the pinion of this wheel interlock with those of the bar. By this means the power is placed in a state of equilibrium with a greater resistance. This case is related to that of toothed wheels, and we will not dilate any more upon that subject.

173. When two powers are in equilibrium by means of a pulley or wheel and axle, it is very evident that we may consider them as though they were in equilibrium at the extremities of a lever whose fulcrum is in the axis of the cylinder or the pulley; hence (151), these powers are to each other reciprocally, as the spaces which they would traverse along their directions, if the equilibrium were disturbed infinitesimally.

Article III.

On the Equilibrium of the Inclined Plane.

Fig. 75.



174. A plane is said to be inclined, when it makes an angle with a horizontal plane ABEF, and this angle is not a right angle.

175. If through any point H, taken upon the line of intersection AB of the inclined plane and the horizontal

plane, two perpendiculars be drawn to this line, the one HK in the horizontal plane, and the other HI in the inclined plane, the angle IHK, formed by these two perpendiculars, is the measure of the inclination of the plane. The plane of the angle IHK, which passes through two lines perpendicular to AB, is perpendicular to the line AB; hence it is perpendicular to the two planes ABCD and ABEF, which intersect in this line; hence this plane is at the same time vertical and perpendicular to the inclined plane.

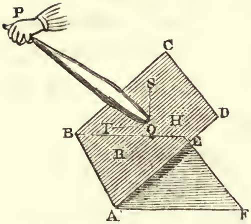
176. Reciprocally, every plane which is at the same time vertical and perpendicular to the inclined plane, is perpendicular to the intersection AB of the inclined plane with the horizontal plane; hence, the lines HI and HK along which it intersects these two last planes, are also perpendicular to AB, and form between them an angle IHK, which is the measure of the inclination of the plane ABCD with reference to the horizontal plane.

177. If through the point I there be drawn a vertical line IL, this line will not leave the plane of the angle IHK; it will meet the horizontal line HK to which it will be perpendicular, and it will form a rectangular triangle IHL. The hypotenuse of this triangle is named the *length* of the inclined plane; the side IL is its *height*, and the other side HL is its *base*.

I.

178. When a body, which, in a single point Q, touches an immoveable plane ABCD, is pushed by a single force P, whose direction PQ, 1st, passes through the point of

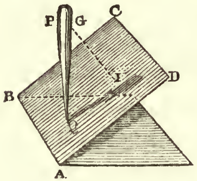
Fig. 76.



contact Q, 2d, is perpendicular to the plane, this body remains at rest.

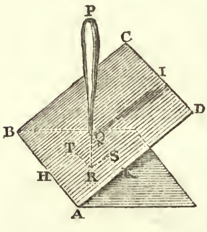
Thus, we can conceive the force P to be applied to the point Q of its direction. Now this direction being perpendicular to the plane, and consequently to all the lines QR, QS, QT, which we can draw in the plane through the point Q, it is similarly disposed with reference to all these lines; there is then no reason why the point Q should move along one of the lines rather than along any other; hence this point, and consequently the whole body, will remain at rest.

Fig. 77.



179. Both the conditions just mentioned are necessary to the rest of the body: for, 1st, if the direction PI of the force does not pass through the point of contact, nothing prevents the point G of the body which is upon this direction from approaching the plane, and the body would be put in motion. 2d. If the direction PQ of the force passes through the point

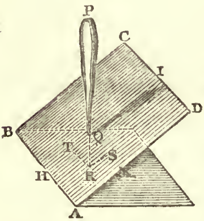
Fig. 78.



of contact, and is not perpendicular to the plane, by conceiving this force to be still applied to the point Q, let its direction be prolonged to QR, and through the point Q draw the line QS perpendicular to the plane; through the two lines QR, QS draw a plane, which will intersect the first plane ABCD somewhere in a line HI, which will pass through

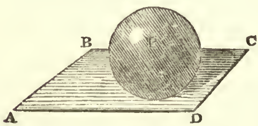
the point Q. This being done, if we represent the force P by the part QR of its direction, and complete the parallelogram QTRS, instead of the force P we may take the two forces represented by QS and QT. Now the force QS, which passes through the point of contact, and whose direction is perpendicular to the plane ABCD, will be destroyed by the resistance of this plane (178); but nothing will oppose the action of the force QT, whose direction is parallel to the plane; the point Q will then move in the direction QH, and the body will not be at rest.

Fig. 78.



180. Hence it follows, that, when a body, affected by the simple action of its gravity, remains in equilibrium upon a plane ABCD, which it touches in a single point Q, 1st, the centre of gravity P of this body, and the point of contact Q, are in the same vertical line; 2d, the plane ABCD is horizontal; for, since the weight of the body can be regarded as a single force applied to its centre of gravity, the body cannot be at rest, unless the direction of this force passes through the point of contact and is perpendicular to the plane upon which the body is sustained.

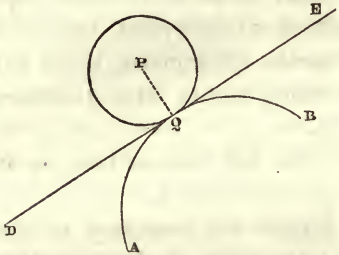
Fig. 79.



181. It follows again, that when a body, affected by the single action of gravitation, is sustained upon an inclined plane ABCD by a single point Q, and this point is found in the vertical drawn through the centre of gravity, this body must tend to slip upon the plane; and the direction QH along

which the point Q tends to move, is the intersection of the plane ABCD with the plane IHK, which is at the same time vertical and perpendicular to the inclined plane.

Fig. 80.



182. What has just been said of a body pushed by a single force against a plane, must apply to a body pushed against a curved surface AQB, which it touches only in the single point Q: that is to say, this body cannot be at rest, unless the direction of the force which pushes it passes through the point Q, and is perpendicular to the curved surface at this point; for this body may be considered as sustained upon the plane DE tangent to the curved surface at the point Q.

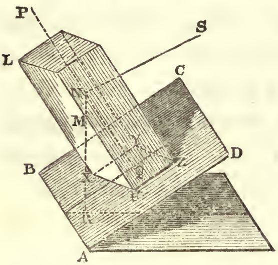
183. We see, then, that when a lever may slide upon its fulcrum, it is not sufficient for its remaining at rest, that the resultant of the two powers which are applied to this lever are directed towards the fulcrum; it is besides necessary that the direction of this resultant should be perpendicular to the surface of the lever at the point where it touches the fulcrum.

II.

184. When a body, pushed by a single force P against an immoveable plane ABCD, is sustained upon this plane by a definite base VXYZ, if the direction PQ

of the force meets the base somewhere in a point Q, and if it is at the same time perpendicular to the plane, the body is at rest; for we have seen (178) that if the base were reduced to the single point Q, rest would take place; it is evident that the other points of support, which the base presents, cannot disturb it.

Fig. 81.

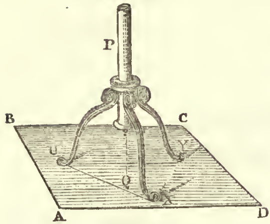


We will demonstrate, as in No. 179, that both these conditions are necessary to the body's remaining at rest: the effect of the first is to prevent the body from turning upon one of the sides of its base; the effect of the second is to prevent it from slipping upon the plane ABCD.

185. If the body, instead of being supported upon the plane by a continuous base, simply touches it by several points separated from each other, we may regard these points as the summits of the angles of a polygonal base; and the body is at rest when the direction of the force, which pushes it against the plane, is perpendicular to this plane, and at the same time passes through the interior of the polygon.

Thus, since the weight of a body may be regarded as a vertical force applied to its centre of gravity P, we see, 1st, that a body, which rests with its base upon a horizontal plane ABCD, cannot be stable, unless the vertical PQ, drawn through the centre of gravity, meets some point Q of the base; 2d, if the body rests upon the plane by a certain number of points of support,

Fig. 82.



U, X, Y, . . . it cannot be stable, unless the vertical PQ, drawn through its centre of gravity F, passes through a point Q taken in the interior of the polygon UXY, which may be formed by joining the exterior points of support by the lines UX, XY, YU.

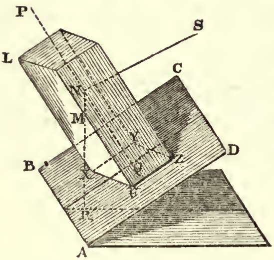
III.

186. Hitherto we have supposed that the body, supported upon a plane, was pushed by a single force ; but it is evident that, if the body is pushed by several forces at the same time, it cannot be at rest, unless the resultant of all these forces satisfies the preceding conditions ; that is to say, unless the direction of this resultant is perpendicular to the plane, and passes through the base of the body. There is then in this case a third condition necessary to the rest of the body, and this condition is, that all the forces which act upon it must have one resultant.

The whole theory of the equilibrium of a body, pushed by as many forces as we please to assume, and supported upon a single resisting plane, consists in the search for the directions and intensities which the forces must have, so that the three conditions just mentioned may be fulfilled. We will content ourselves with developing it for a few simple cases, and principally for that in which the body is pushed by two forces.

IV.

Fig. 81.



187. Let LUZ be a body supported by a base UXYZ upon a resisting plane ABCD, and solicited at the same time by two forces R, S. According to the preceding remarks, in order that the body may be at rest, it is necessary, 1st, that the two forces R, S should have one resultant: now, two forces cannot have one resultant, unless their directions are in the same plane, (10); hence, 1st, the directions of the two forces R, S must be included in the same plane, and intersect in a certain point N.

2d. It is necessary that the direction PNQ of the resultant of the two forces R, S, should be perpendicular to the plane ABCD: now the resultant of the two forces is always contained in the plane drawn through their directions; hence, 2d, the plane in which the two forces R, S are directed, should be perpendicular to the plane ABCD.

3d. It is necessary that the direction of the resultant should pass through a point Q of the base.

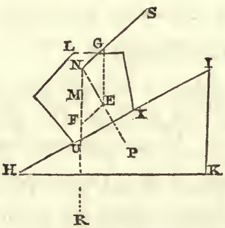
188. From this it follows, that if one of the two forces, for example the force R, is the weight of the body which may be considered as applied to the centre of gravity M, and whose direction MR is vertical, the body cannot be at rest upon an inclined plane ABCD, unless the direction NS of the other force is contained

in a vertical plane, drawn through the centre of gravity M, perpendicularly to the inclined plane, and, moreover, that the direction PQ of the resultant of the two forces is perpendicular to the inclined plane, and passes through a point Q of the base of the body.

Now all these conditions, relative to the directions of the forces, being supposed to be filled, we will seek the relations which the two forces R, S, and the weight P of the plane, have for each other in the case of equilibrium.

V.

Fig. 83.



189. Let LXU be a body supported by its base UX upon a resisting plane HI, and kept at rest upon this plane by the two forces R, S. Having prolonged the directions of the two forces until they meet in a point N, draw through this point the line NP perpendicular to the plane HI. We have seen that this line will be the direction of the resultant of the two forces R, S. Hence, if we represent this resultant by the part NE of its direction, and if, by drawing through the point E the lines EG, EF, parallel to the directions of the forces R, S, we complete the parallelogram NFEG, the sides NF, NG will represent the intensities of the forces R, S. Hence, by naming P the weight on the plane which is equal to the resultant, we shall have

R : S : P : : NF : NG or FE : NE.

In order to have the ratios of these three forces expressed in quantities independent of the construction of the parallelogram NFEG, we will remark, that in the triangle NEF the sides are to each other in the ratio of the sines of the opposite angles, or we shall have

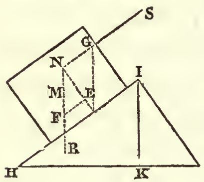
NF : FE : EN : : sin NEF : sin FNE : sin NFE.

Hence we shall have

R : S : P : : sin NEF : sin FNE : sin NFE.

Now these three angles are those which form with each other the directions of the three forces R, S, P; hence these forces are to each other as the sine of the angle which the directions of the other two forms.

Fig. 84.



We see then that, of the six things which may be considered in this equilibrium, and which are, the directions of the three forces and their intensities, any three being given, we can determine the other three, in all cases where, of the six things which we consider in the triangle NEF, namely, the angles and the sides, the three analogous ones being given, we can determine the three others.

190. If one of the forces, for example the force R, is the weight of the body whose direction is vertical, and passes through the centre of gravity M, and the direction of the force S, which retains the body in equilibrium upon the inclined plane, is parallel to this plane, draw the base KH and the height KI of the inclined

plane; the right-angled triangles NFE, IHK will be similar, because the angles NFE, HIK, whose sides are parallel each to each, will be equal, and we shall have

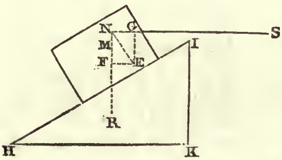
NF : FE : EN : : HI : IK : KH;

hence we shall also have

R : S : P : : HI : IK : KH.

Thus, in this case, the weight of the body is to the force which holds it in equilibrium, as the length of the inclined plane is to its height.

Fig. 85.



191. By supposing still that the force R is the weight of the body, if the direction of the force S is horizontal, and consequently parallel to the base HK of the inclined plane, the right-angled triangles NFE, HKI are again similar; because the sides of the one will be perpendicular to the sides of the other, and we shall have

NF : FE : EN : : HK : KI : IH,

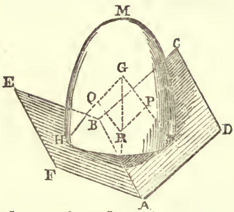
we shall also have

R : S : P : : HK : KI : IH.

Hence, the weight of the body is to the force which holds it in equilibrium, as the base of the inclined plane is to its height.

VI.

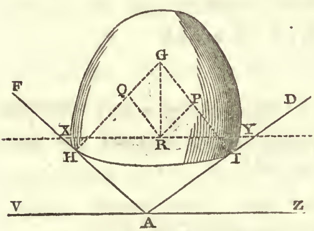
Fig. 86.



192. Let us now consider the equilibrium of a body supported by two inclined planes. Let M be a body subjected to the single action of gravity, and retained by the two inclined planes ABCD, ABEF, which intersect somewhere in the line AB. Let H, I be the points at which the body touches the two planes, and GR the vertical line drawn through its centre of gravity, and which will consequently be the direction of its weight. It is evident that this body cannot be at rest, unless its weight R can be decomposed into two other forces P, Q, applied to the same body, and which are destroyed by the resistance of the two planes: or, what is the same thing, unless the directions of the two forces P, Q pass through the points of support I, H, and are perpendicular each to the corresponding inclined plane. Now the direction of a force and those of its two components are contained in the same plane, and necessarily meet in the same point; hence, in order that the body M may be at rest between the two inclined planes, it is necessary that the perpendiculars IG, HG, drawn through the points of support I, H to the two inclined planes, should be in the same plane with the vertical drawn through the centre of gravity of the body, and intersect this vertical in the same point G.

193. Hence it follows, that, in order that a body M may be at rest between two inclined planes, independently of the position of the body, the planes should satisfy the condition that the line AB of their intersection should be horizontal. Thus, the plane IGH, which must contain the vertical GR, and the perpendiculars IG, HG to the two inclined planes, is at the same time vertical and perpendicular to these two planes; hence, reciprocally, the two inclined planes must be perpendicular to the vertical plane IGH; hence the line AB of their intersection should be perpendicular to this same plane, and consequently horizontal.

Fig. 87.



194. These conditions, which have for object the respective positions of the body and of the two inclined planes, being supposed to be fulfilled, in order to find the ratio of the weight R of the body to the weights P, Q, which the two planes support, we will remark that the plane IGH, vertical and perpendicular to the two inclined planes containing the angles which these two planes form with the horizon, includes all that relates to the question, and we may be content to consider it alone, as in *Fig.* 87. Hence, let AD, AF be the intersections of the vertical plane IGH with the two inclined planes; these lines form with the horizontal line VZ, or with any other horizontal line XY, angles which will measure the inclinations of the two planes. This being granted, if we represent the weight of the

body by the part GR of its direction, and complete the parallelogram GPQR, we shall have

$$R : P : Q :: GR : RQ : QG.$$

Now the triangles GQR, XAY, whose sides are perpendicular each to each, give

$$GR : RQ : QG :: XY : YA : AX;$$

hence we shall also have

$$R : P : Q :: XY : YA : AX;$$

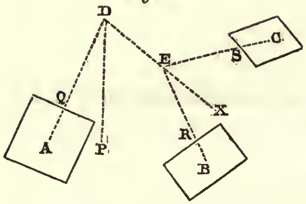
or, lastly, because the sides of the triangle XAY are proportional to the sines of the opposite angles, we shall have

$$R : P : Q :: \sin YAX : \sin AXY : \sin XYA,$$

that is to say, by representing the weight of the body by the sine of the angle which the two inclined planes form with each other, the weights which these two planes support are to each other reciprocally as the sines of the angles which these planes form with the horizon.

VII.

Fig. 88.



195. Lastly, if a body is sustained by three points A, B, C, upon three inclined planes, it is evident that this body cannot be at rest unless its weight P, whose direction DP is vertical, and passes through the centre of gravity of the body, can be decomposed into three other forces Q, R, S, which are destroyed by the resistances of the inclined planes; that is to say, unless the directions of the three forces Q, R, S pass through the three points of support, and are each perpendicular to the corresponding inclined plane.

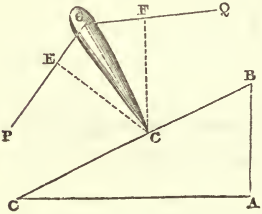
196. In order that the body may be at rest, it is necessary that the weight P should be susceptible of being decomposed into two forces Q, X; the first of which, Q, being directed towards one of the points of support A, perpendicularly to the inclined plane which passes through this point, the other force X may itself be decomposed into two others R, S, in the directions of the two other points of support B, C, perpendicular to the other inclined planes.

Hence we see, in this case, it is not necessary that the perpendiculars to the inclined planes, drawn through the points of contact A, B, C, should all three meet in the same point, nor even that they should all three meet the vertical drawn through the centre of gravity of the body.

THEOREM.

197. *When a body without gravity, supported upon an inclined plane* BC *by a single point* C, *is in equilibrium between two powers* P, Q, *these powers are to each other reciprocally as the spaces which they would traverse in the line of their directions, if the equilibrium were disturbed infinitesimally.*

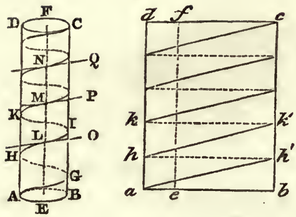
Fig. 89.



DEMONSTRATION. The two powers P, Q being in equilibrium, their resultant should be perpendicular to the inclined plane, and pass through the point of support C (186); it follows from this that the directions of these powers should coincide in a certain point G, and that the line GC should be perpendicular to the inclined plane. This being established, from the point C draw upon the directions of the powers P, Q, the perpendiculars CE, CF: it is evident that the angle ECF, supposed to be invariable, may be considered as a bent lever, at the extremities of which are applied the two powers in equilibrium about the point of support C: hence we can demonstrate, as in No. 151, that the powers are to each other reciprocally as the spaces which they would traverse in the line of their directions, if the equilibrium were disturbed infinitesimally.

On the Screw.

Fig. 90.



198. If we conceive a cylinder ABCD to be enveloped by a thread AGHIK . . . ., and so disposed that the angles FLO, FMP, FNQ . . . ., formed by the direction of the thread with the lines drawn upon the surface of the cylinder parallel to the axis, are equal to each other, the curve which the thread traces upon the surface of the cylinder is named a *helix*.

199. Hence it follows, that if we develope the surface of the cylinder, and extend it upon a plane, as we see in *abcd*, 1st, the developement *ah'* or *hk'* of one revolution of the helix will be a straight line; because the angles which this line will form with all such lines as *ef*, drawn parallel to the side *ad*, will be equal to each other. 2d. This developement *ah'* of a revolution of the helix will be the hypotenuse of a right-angled triangle *abh'*, whose base *ab* will be equal to the circumference of the base of the cylinder, and whose height *bh'* will be equal to the distance of the revolution which we consider, from the revolution which follows it. 3d. All the hypotenuses *ah'*, *hk'* being parallel to each other, the right-angled triangles *abh'*, *hh'k'* . . ., will be equal and similar, and will have equal heights. Hence the intervals LM, MN, . . . between two consecutive revolutions of the helix, considered upon the surface of the cylinder, are everywhere equal to each other.

Fig. 91.

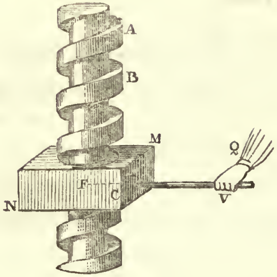
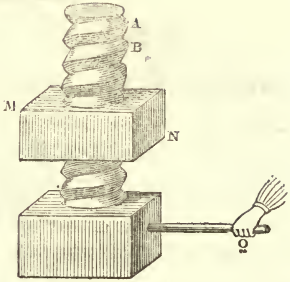


Fig. 92.



200. This being established, the screw may be considered as a right cylinder, enveloped by a projecting fillet, adhering and wound as a helix upon the surface of the cylinder. In the wooden screw, the form of the fillet is such, that if it be cut by a plane drawn through the axis of the cylinder, its section is most frequently an isosceles triangle, as may be seen in *Fig.* 92; but in large iron screws which are made with care, the section of the fillet is rectangular, as in *Fig.* 91. The constant interval AB, which is found between two consecutive revolutions of the fillet, is named *the pitch of the screw*, or the *helical interval.*

201. The piece MN, into which the screw enters, is named the *nut.* Its cavity is invested with another projecting fillet, adhering, and wound likewise as a helix; and whose figure is such that it exactly fills the intervals left by the fillets of the screw. Thus, the screw can turn in its nut, but it cannot do so without moving in the direction of its axis; and for one entire revolution, it moves in the direction of the axis by a quantity equal to the pitch of the screw.

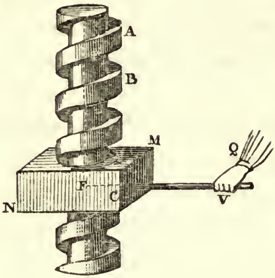
202. Sometimes the screw is fixed, and the nut moves around it; then, for each revolution, the nut is carried upon the screw by a quantity equal to the interval.

203. The screw may serve to elevate weights or overcome resistances; but it is employed most generally when great pressures are proposed to be exerted. For this purpose we apply a power Q to the extremity of a bar which traverses the head of the screw, *Fig.* 92, or the nut, *Fig.* 91, according as it is the one or the other of these two pieces which is moveable; and this power, by causing the piece to turn to which it is applied, makes the head of the screw advance towards the nut, or reciprocally. The objects to be compressed are ranged between two plates; one of these plates is fixed, the other is pressed by the movable piece, which can advance only by reducing the volume of the objects.

We now propose, disregarding the friction, to find the ratio of the power Q to the resistance P, which is in equilibrium with it by being opposed to the motion of the moveable piece, along *a direction parallel to the axis of the screw;* and, because the effect is absolutely the same, whether the screw turns in its nut, or the nut turns upon the screw, it will be sufficient to examine the latter case.

204. The screw being fixed and in a vertical position, we will conceive the nut to be left to the action of gravity, and even, if we please, that it is charged with an additional weight; it is evident that it will descend by turning, and that it will traverse all the interior fillets of the screw, by sliding over them as over inclined surfaces. It is also evident that we oppose this effect

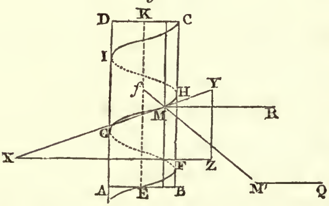
Fig. 91.



by preventing the nut from turning around the screw, and consequently by applying to the extremity of the bar FV a power Q which is directed perpendicularly to this bar, and in a plane perpendicular to the axis of the screw.

205. Suppose, for an instant, that the nut rests upon the surface of the fillet of the screw only in a single point; this point, during the motion of the nut, will describe a helix, whose pitch will be the same as that of the screw, and which we may conceive to be traced upon the surface of a cylinder, whose radius would be equal to the distance of the describing point from the axis of the screw. Let ABCD be the cylinder, EFGHI the helix

Fig. 93.



in question, and M the point of the nut which describes it. Let XY be the tangent of the helix at the point M; through a point Y of this tangent draw the vertical YZ, equal to the pitch of the helix, and in the vertical plane XYZ of the horizontal line XZ: it is clear, that this last line will be equal to the circumference of the base of the cylinder ABCD (199), or to the circumference of the circle whose radius is CK; and we will represent it by circ. CK.

This being established, the point M, which we may regard as charged with the whole weight P of the screw, will be supported upon the helix as though it were upon

the inclined plane XY; thus, in order to hold it in equilibrium by means of a force R, which should be immediately applied to it, and which should be directed parallel to XZ, it is necessary (191) that this force R should be to the weight P, as the height of the inclined plane is to its base, or that we should have,

$$R : P :: YZ : \text{circ. } CK.$$

But if, instead of a force R immediately applied to the point M, we employ a force Q, whose direction is parallel to that of the first, and which acts at the extremity of a bar $M'f$, it is necessary that this force should exert upon the point M the same effect as the force R, and for that purpose, these forces should be to each other reciprocally as their distances from the axis of the cylinder; that is to say, we should have,

$$Q : R :: Mf : M'f;$$

or, since the circumferences of circles are to each other as their radii, we should have,

$$Q : R :: \text{circ. } Mf : \text{circ. } M'f.$$

Then, by multiplying together in order this proportion and the first, we shall have, since circ. CK=circ. Mf,

$$Q : P :: YZ : \text{circ. } M'f;$$

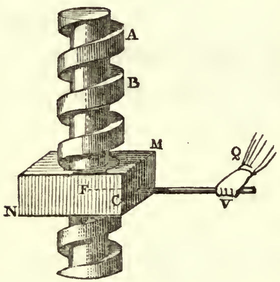
that is to say, the power which retains the nut in equilibrium, will be to the weight of the nut, as the pitch of

the screw is to the circumference of the circle which the power tends to describe.*

206. Since the distance of the point M from the axis of the screw does not enter into this proportion, it follows that, whatever may be this distance, the ratio of the weight P of the nut to the power Q, which is in equilibrium with it, is always the same, provided this power is always applied to the same point.

207. If the fillet of the nut is supported upon that of the screw by several points unequally distant from the axis of the screw, as it generally happens, the total weight of the nut can be regarded as divided into partial weights, each applied to one of the points of support. Now the partial power applied to the point V, and which makes an equilibrium with one of these partial weights, is to this weight, in the constant ratio of the pitch of the screw, to the circumference which the power tends to describe.

Fig. 91.



Hence the sum of the partial weights, or the total weight of the nut, is to the sum of the partial powers, or to the total power Q, in the same ratio.

208. From this it follows: 1st. The force necessary to be applied to the nut parallel to the axis of the screw, to produce an equilibrium with the power Q, which tends

* This relation is independent of the form of the fillet of the screw; because the only question is to prevent the moveable piece from turning around a line which is supposed to be fixed, No. 203.

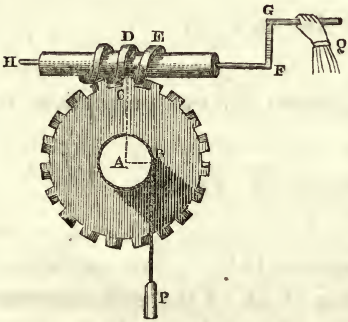
to turn the nut, should be to this power, as the circumference of the circle which this power tends to describe, is to the pitch of the screw;

2d. For the same screw, the effect of the power Q is as much greater as this power is applied at a greater distance from the axis of the screw;

3d. For two different screws, the power being applied at the same distance from the axis, its effect is as much more considerable as the height of the pitch is less; that is to say, the closer the fillets of the screw are together, the greater is the effect of the power for compressing in the direction of the axis.

II.

Fig. 94.



209. The screw is sometimes employed for communicating to a toothed wheel a motion of rotation upon its axis. For this purpose, having given to the screw a pitch DE, equal to one of the divisions of the toothed wheel, it is so arranged that its axis is in the plane of the wheel, and its fillet catches in the teeth. This being done, when a power Q turns the screw upon its axis, by means of a crank FG, the fillet carries along the teeth, which follow each other, and it turns the wheel in spite of the resistance P, which opposes its rotary motion.

When the screw is employed for this purpose, it is named the *endless screw.*

To find the ratio of the power Q to the resistance P in the case of equilibrium, suppose the resistance is suspended to a weight by a cord which envelopes the arbor of the wheel. By virtue of this weight, the tooth of the wheel presses the fillet of the screw parallel to the axis HF; and if we name R this pressure, we have (164),

$$P : R :: AC : AB.$$

Now we may regard the pressure of the tooth as that which a nut, pushed by a force R parallel to the axis of the screw, would exert; we have, in the case of equilibrium,

$$R : Q :: \text{circ. } FG : DE;$$

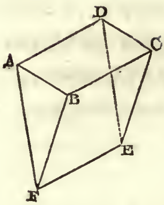
hence, by multiplying together the two proportions in order, we have,

$$P : Q :: AC \times \text{circ. } FG : AB \times DE;$$

that is to say, the resistance is to the power, as the product of the radius of the wheel by the circumference which the crank describes, is to the product of the radius of the arbor of the wheel by the pitch of the screw.

On the Wedge.

Fig. 95.

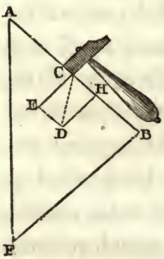


210. The wedge is a triangular prism ABCDEF, which is introduced by its sharp edge EF into a crack, to split or separate the two parts of a body. It is also made use of for exerting great pressures or to stretch cords.

Knives, hatchets, punches, and, in general, all cutting and penetrating instruments may be considered as wedges.

The face ABCD, upon which we strike the wedge to sink it, and which receives the action of the power, is named the *heel of the wedge;* the side EF, by which the wedge commences to penetrate, is named the *blade;* and the name of *sides* are given to the faces AFED, BFEC, by which it compresses the bodies which it has to separate; or, since we are accustomed to represent the wedge by its triangular profile ABF, the base AB of the triangle is called the heel of the wedge; AF and BF are its sides.

Fig. 96.

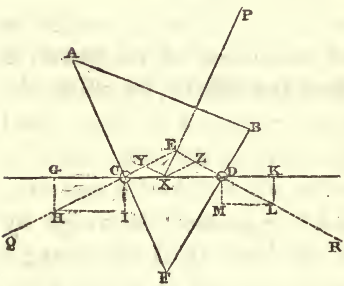


211. We are accustomed to suppose that the direction of the power is perpendicular to the heel of the wedge; because, generally, the wedge is sunk by striking upon the heel with a hammer, or with any other object which has no connection with it, and which, in this case, if the direction CD of the shock is not perpendicular to the surface of the heel, the action is naturally decomposed into two other forces, CH, CE; the first of which, being parallel to the

heel of the wedge, can have no other effect than that of making the hammer slip, and the second being perpendicular to the face AB, is the only one which is transmitted to the wedge, and conduces to the effect which is to be produced. But if the power were applied to the wedge by means of a cord, whose point of attachment could not slip, then, by considering this power, it would be necessary to take account of its direction.

I.

Fig. 97.



212. Let C, D, be two points separated by a wedge ABF, and retained by a cord CD, which is attached to them, and which opposes their separation; suppose, moreover, that these points are supported against a resisting plane which prevents them from moving in a direction perpendicular to the cord, and that a power P, applied perpendicularly to the heel AB of the wedge, makes an effort to separate them and move them in the direction of the length of this cord. This being established, it is required to determine, 1st, the tension which results in the cord CD; 2d, the force necessary to be applied to one of the two points C, D, to prevent them from moving in the direction of the cord; 3d, the pressure which each of these points exerts upon the plane which resists them.

It must be remarked first, that, if the direction of the power P is not such that it may be decomposed into two others Q, R, whose directions pass through the points C, D, and are perpendicular to the sides AF, BF, the wedge will turn between the two points C, D until this condition is fulfilled, and then only will the power P produce all its effect. We will suppose, moreover, that, having drawn through the points C, D the lines CE, DE perpendicular to the sides of the wedge, the point of intersection E of these two lines is in the direction of the power P.

This being the case, the force P will be decomposed into two other forces Q, R, directed along EC, ED; and, if we represent this force by the part EX of its direction, and finish the parallelogram EZXY, we shall have,

$$P : Q : R :: EX : EY : EZ \text{ or } YX;$$

or, since the two triangles EXY and ABF, whose sides are perpendicular each to each, are similar, we shall have,

$$P : Q : R :: AB : AF : BF,$$

and, consequently,

$$Q = \frac{P \times AF}{AB}, \qquad R = \frac{P \times BF}{AB}.$$

The point C not being able to move in the direction ECH, because of the resistance of the plane upon which it is supported, the force Q, which is applied to it, will be decomposed into two others; one of which, in the

direction of the line CI, perpendicular to the cord, will be destroyed by the resistance of the plane, and the other, in the direction of the prolongation of DC, will be employed in stretching the cord. Thus, by making CH=EY, and completing the rectangle CGHI, the two components of the force Q will be represented by CI and CG; and we shall have,

$$\text{Q} : \text{force CI} : \text{force CG} :: \text{CH} : \text{CI} : \text{CG},$$

and, consequently,

$$\text{force CI}=\frac{\text{Q}\times\text{CI}}{\text{CH}},$$

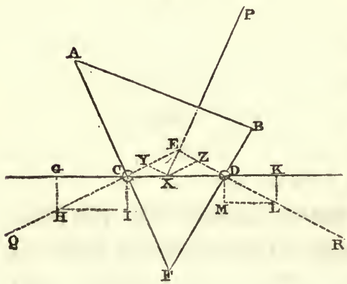
$$\text{force CG}=\frac{\text{Q}\times\text{CG}}{\text{CH}};$$

or, substituting for Q its value previously found, we shall have,

$$\text{force CI}=\frac{\text{P}\times\text{AF}\times\text{CI}}{\text{AB}\times\text{CH}},$$

$$\text{force CG}=\frac{\text{P}\times\text{AF}\times\text{CG}}{\text{AB}\times\text{CH}}.$$

Fig. 97.



In like manner, if, upon the prolongation of ED, we make DL=EZ, and complete the rectangle DKLM, whose side DK is upon the prolongation of CD, and whose side DM is perpendicular to CD, the force R will be decomposed into two

others DM, DK, the first of which will be destroyed by the resistance of the plane, and the second will be wholly employed in acting upon the cord; and we will find, likewise,

$$\text{force DM} = \frac{\text{R} \times \text{DM}}{\text{DL}} = \frac{\text{P} \times \text{BF} \times \text{DM}}{\text{AB} \times \text{DL}},$$

$$\text{force DK} = \frac{\text{R} \times \text{DK}}{\text{DL}} = \frac{\text{P} \times \text{BF} \times \text{DK}}{\text{AB} \times \text{DL}}.$$

Thus the cord CD will be drawn in one direction by the force CG and in the contrary direction by the force DK.

Now, when a cord is drawn in contrary directions by two unequal forces, the tension which it suffers is always equal to the smaller of these two forces; for, when the two forces are equal, one of them is the measure of the tension of the cord; and when they are unequal, the excess of the greater over the smaller, not being counter-balanced by anything, does not contribute to stretch the cord, and has no other effect than to draw it along in the direction of its length.

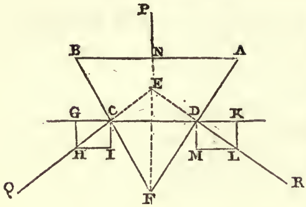
Hence, 1st, the tension of the cord CD will be equal to the smallest of the two forces CG, DK.

2d. The cord will be drawn along in the direction of its length and in that of the greater of the two forces CG, DK; so that, in order to oppose this motion, it will be necessary to apply to one of the two points C, D a force equal to the difference of these two forces, and directly opposed to the greater.

3d. The pressures exerted by the two points C, D upon the plain which retains them, will be equal; the first to the force CI, the second to the force DM.

II.

Fig. 98.



213. If the sides AF, BF of the wedge be equal, the heel AB is parallel to the cord which retains the two points C, D; and, at the same time, if the direction of the force P is perpendicular to the middle of AB, 1st, the wedge will not turn, because the lines CE, DE, drawn through the two points of support perpendicular to the sides of the wedge, will meet in a point E of the direction of the power. 2d. Both sides being perfectly alike, the forces CG, DK will be equal to each other, and each of them will be the measure of the tension of the cord CD. 3d. By letting fall from the point F the perpendicular FN upon the heel of the wedge, the two triangles CGH, BNF, whose sides will be perpendicular each to each, will be similar, and give,

$$CH : CG :: BF : FN.$$

Now we have,

$$Q : \text{force } CG :: CH : CG,$$

and, consequently,

$$Q : \text{force } CG :: BF : FN;$$

But, we have, also,

$$P : Q :: AB : BF.$$

Hence, by multiplying together these two proportions in order, we shall have

$$\text{P : force CG :: AB : FN};$$

that is to say, in this case, the power P will be to the tension of the cord CD, as the heel of the wedge is to its height.

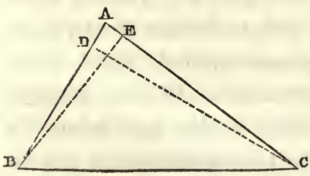
We will not make any application of the theory of the wedge to the use which may be made of this instrument for splitting bodies; because, under such circumstances, the resistance which has to be overcome is always unknown, and it is useless to know the ratio of this resistance to the power which is in equilibrium with it.

LEMMA.

214. *If from the summits* B, C *of two of the angles of a triangle, the perpendiculars* BE, CD *be dropped upon the opposite sides, these perpendiculars will be to each other reciprocally as the sides upon which they are dropped; that is to say, we shall have,*

$$\text{BE : CD :: AB : AC.}$$

Fig. 99.



Demonstration. By considering AB as the base of the triangle, the perpendicular CD will be its height, and the surface of the triangle will be $\frac{AB \times CD}{2}$. In

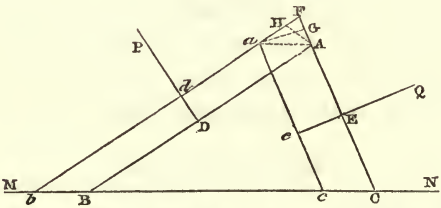
like manner, by taking AC for the base, the surface will be $\frac{AC \times BE}{2}$; hence, we shall have the two equal products $AC \times BE = AB \times CD$, which will give the proportion,

$$BE : CD :: AB : AC.$$

THEOREM.

215. *When the two powers* P, Q *are applied to the faces* AB, AC *of a wedge, the third face of which* BC *is supported upon a resisting plane* MN, *and these two powers are in equilibrium through the resistance of the plane, they are to each other reciprocally as the spaces which they traverse in the line of their directions, if the equilibrium be disturbed infinitesimally.*

Fig. 100.



DEMONSTRATION. Since the powers P, Q are in equilibrium, their directions are perpendicular to the faces of the wedge to which they are applied (179). Now suppose, by virtue of a derangement in the equilibrium, the wedge slips upon the resisting plane, and takes the infinitely near position *abc*; and the direction QE is prolonged until it meets *ac* in *e*: it is evident that the

small lines Ee, Dd will be the spaces which the powers will have traversed in the lines of their directions.

Lastly, draw Aa; prolong CA, ba until they intersect somewhere in F, and from the points A, a let fall the perpendiculars AH, aG; we will have evidently Ee=Ga and Dd=AH.

This being determined, the powers P, Q being in equilibrium, they are to each other as the sides of the wedge to which they are applied; then we shall have (212),

$$\text{P} : \text{Q} :: \text{AB} : \text{AC},$$

and, because of the similar triangles ABC, FaA,

$$\text{P} : \text{Q} :: \text{F}a : \text{FA};$$

hence (214), we shall have,

$$\text{P} : \text{Q} :: a\text{G} : \text{AH}$$

and, consequently,

$$\text{P} : \text{Q} :: \text{E}e : \text{D}d.$$

216. We have seen that the analogous proposition takes place in the equilibrium of all the machines which we have considered. By following the same process, it could be demonstrated directly that, when two powers are in equilibrium by means of points of support which any machine presents, they are to each other reciprocally as the spaces which they would traverse in the lines of their directions, if the equilibrium were infinitesimally

disturbed. By means of this proposition, it will be easy to find in practice the relation which should subsist between a power and a resistance applied to a proposed machine, in order that these forces should be in equilibrium, leaving out of consideration friction and other obstacles to motion.

NOTE.

A NEW DEMONSTRATION OF THE PARALLELOGRAM OF FORCES.

BY M. CAUCHY.*

If we suppose, as in general, a force to be represented by a length laid off from its point of application along the direction in which it acts, the resistance R of two forces P, Q, simultaneously applied to a material point (A), will be represented in intensity and direction by the diagonal of the parallelogram constructed upon these two forces. This proposition has already been demonstrated in several manners. But, among the demonstrations which have been given, some require the consideration of new material points connected with the point (A) by rigid and invariable lines; others are founded upon the use of the differential calculus, or of derived functions; others again are deduced from the relations which exist between the cosines of certain angles. I am here about to demonstrate the same proposition without recurring to these different considerations,

* This demonstration is extracted from the work which M. Cauchy publishes in numbers under the name of *Exercises de Mathematiques*.

and, in order to attain my object, I will establish successively several lemmas, which may be enunciated as follows.

LEMMA I.

If we designate by R *the resultant of the two forces* P, Q, *simultaneously applied to the point* (*a*), *and by* x *any number, the resultant of two forces, equal to the products* Px, Qx, *and directed along the same lines as the forces* P, Q, *will be represented by the product* Rx, *and directed along the same line as the force* R.

DEMONSTRATION. In the first place let $x=\frac{m}{n}$, m and n representing any two whole numbers. We shall have

$$\text{P}x=m\frac{\text{P}}{n}, \qquad \text{Q}x=m\frac{\text{Q}}{n}.$$

Besides, we may consider the component P, or $m\ \frac{\text{P}}{n}$, as produced by the addition of several forces equal to $\frac{\text{P}}{n}$, and the component Q, or $m\ \frac{\text{Q}}{n}$, as produced by the addition of as many forces equal to $\frac{\text{Q}}{n}$. From this it is easy to conclude, that the resultant of the forces Px, Qx, will both be produced by the addition of several forces equal to the resultant of $\frac{\text{P}}{n}$ and $\frac{\text{Q}}{n}$. Moreover, it is evident that the first two resultants are to the last, as the

numbers n and m are to unity. Hence, the second resultant will be equivalent to the first multiplied by the ratio $\frac{m}{n}=x$, that is to say, to the product Rx.

Let us suppose in the second place that the number x is irrational. Then we can vary the whole numbers m and n in such a way that the fraction $\frac{m}{n}$ will converge to the limit x; and it is evident that, in this case, the resultant of the forces $\frac{m\text{P}}{n}$, $\frac{m\text{Q}}{n}$, always directed along the same line, and always equal to $\frac{m\text{R}}{n}$, will tend more and more to coincide, in intensity and direction, with the resultant of the forces Px, Qx. Hence, this last resultant will be directed along the same line as the force R, and it will have for measure the limit of the product $\frac{m\text{R}}{n}$, that is to say, the product Rx.

Corollary.

If we designate by the notation $(\widehat{\text{P}, \text{Q}})$ the angle included between the directions of the two forces P and Q $(\widehat{\text{P}, \text{R}})$, $(\widehat{\text{Q}, \text{R}})$ will be the angles included between the directions of the components P and Q and their resultant R. This being fixed, if we make successively Rx=P, Rx=Q, or, what amounts to the same, $x=\frac{\text{P}}{\text{R}}$, $x=\frac{\text{Q}}{\text{R}}$, we

will conclude from the preceding lemma, 1st, that the force P may be replaced by two components $\frac{P^2}{R}$ and $\frac{PQ}{R}$, which form with it angles equal to $(\widehat{P, R})$ and $(\widehat{Q, R})$; 2d, that the force Q may be replaced by two components $\frac{Q^2}{R}$ and $\frac{PQ}{R}$, which form with it the angles $(\widehat{Q, R})$ and $(\widehat{P, R})$.

LEMMA II.

The resultant R *of two forces* P, Q, *which intersect at right angles, is represented in intensity by the diagonal of the triangle constructed upon the two components, so that we have,*

$$R^2 = P^2 + Q^2 \qquad (1)$$

Demonstration. Let us conceive the force P to be replaced by the two above-mentioned components, that is to say, by two forces $\frac{P^2}{R}$ and $\frac{PQ}{R}$, which form with it the angles $(\widehat{P, R})$ and $(\widehat{Q, R})$. Let us conceive also that the force Q is replaced by two components $\frac{Q^2}{R}$ and $\frac{PQ}{R}$, which form with it the angles $(\widehat{Q, R})$ and $(\widehat{P, R})$. We

can suppose that the forces $\frac{P^2}{R}$, $\frac{Q^2}{R}$ are directed along the same line as the resultant R, and then the two forces equivalent to $\frac{PQ}{R}$ will each form with the direction of R an angle equal to $(\widehat{P, Q})$. Hence, they will form between them an angle equal to double $(\widehat{P, Q})$. Hence, since the angle $(\widehat{P, Q})$ is a right angle, by hypothesis, the forces equivalent to $\frac{PQ}{R}$ will be equal, but directly opposed. Consequently, they will be in equilibrium; and for the forces $\frac{P^2}{R}$, $\frac{Q^2}{R}$, directed along the same line, we can substitute only the forces P, Q, or their resultant R. Hence, we shall have the equation,

$$R=\frac{P^2}{R}+\frac{Q^2}{R},$$

from which formula (1) is immediately deduced.

This demonstration is due to Daniel Bernoulli.

LEMMA III.

The resultant R *of the two forces* P, Q, *which intersect at right angles, is represented, not only in intensity, as we have proved above, but also in direction, by the diagonal of the rectangle constructed upon the two components.*

DEMONSTRATION. This proposition is evident in the case where the forces P, Q are equal to each other.

Then the resultant R should necessarily divide the angle $\widehat{(P, Q)}$ into two equal parts, and we have, by virtue of Lemma II,

$$R^2=2P^2, \text{ or } R=P\sqrt{2}.$$

It is also easy to demonstrate, Lemma III, in the case where we suppose $Q^2=2P^2$, or $Q=P\sqrt{2}$. Thus, we will consider three forces, equal to P, directed along three lines which are perpendicular to each other. These three forces will be represented by three sides of a cube which meet in the same summit. Moreover, the resultant of two of these forces being equal to $P\sqrt{2}$, and directed along the diagonal of one of the forces of the cube, the resultant Q of the three forces will of necessity be included in the whole plane, which will contain one of the forces P and the diagonal of the square constructed upon the other two. Now, there are three planes of this kind, and these three planes intersect in the diagonal of the cube. Hence, the resultant of the three forces P, or what is the same, the resultant of the forces P and $P\sqrt{2}$, which intersect at right angles, will be directed along the diagonal of the cube, which is at the same time the diagonal of the rectangle constructed upon the forces P and $P\sqrt{2}$.

It might be proved, precisely in the same manner, that if we designate by m a whole number, and suppose Lemma III to be demonstrated in the case where we have $Q=P\sqrt{m}$, the resultant of three forces, respectively equivalent to

$$P, P, P\sqrt{m},$$

and represented by three lines perpendicular to each other, will be directed along the diagonal of the rectangular parallelopipedon, which will have these same lines for sides. From this we conclude, in the admitted hypothesis, that Lemma III will also subsist, if we take for Q the resultant of the forces P and $P\sqrt{m}$, that is to say, if we make $Q = P\sqrt{m+1}$. Besides, Lemma III is evident, when we have $Q = P$, or, what is the same, $m=1$. Hence, this lemma will also subsist, if we take

$$Q = P\sqrt{1+1} = P\sqrt{2}, \text{ or } Q = P\sqrt{2+1} = P\sqrt{3}, \text{ etc.,}$$

or, in general, $Q = P\sqrt{m}$, m being any whole number.

Let us conceive now, that m and n designate two whole numbers; and construct a rectangular parallelopipedon, which has for its sides three lines representing the three forces

$$P,\ P\sqrt{m},\ P\sqrt{n}.$$

The resultant of these three forces will evidently be contained; 1st, in the plane which includes the force $P\sqrt{n}$ and the diagonal $P\sqrt{m+1}$ of the rectangle constructed upon the forces P, $P\sqrt{m}$; 2d, in the plane which includes the force $P\sqrt{m}$ and the diagonal $P\sqrt{n+1}$ of the rectangle constructed upon the forces P, $P\sqrt{n}$. Hence, this resultant will be directed along the diagonal of the parallelopipedon; and the plane, which contains the same resultant with the force P, will intersect the plane of the two forces $P\sqrt{m}$, $P\sqrt{n}$ along the diagonal of the rectangle constructed upon these two forces. Hence the resultant of the forces $P\sqrt{m}$, $P\sqrt{n}$, which evidently should be comprised in the plane in question,

will be directed along this last diagonal. Hence Lemma III will subsist, when we replace the forces P and Q by two forces equal to $Q\sqrt{m}$, $P\sqrt{n}$, that is to say, by two forces whose squares are to each other in the ratio of m to n. Hence, Lemma III will also subsist between the forces P and Q, if we suppose

$$\frac{Q^2}{P^2}=\frac{m}{n} \text{ or } Q=P\sqrt{\frac{m}{n}}.$$

Now, let $Q=Px$, x designating any number whatever. We can vary the whole numbers m and n, in such a manner that the ratio $\frac{m}{n}$ will converge to the limit x^2, and it is evident, that, in this case, the resultant of the forces P and $P\sqrt{\frac{m}{n}}$, directed along two lines perpendicular to each other, will tend more and more to coincide, in intensity and direction, on the one hand with the resultant of the forces P, Px, and, on the other, with the diagonal of the rectangle constructed upon these two forces. Hence, the resultant of the forces P, Px will be represented by the diagonal in question.

Corollary I.

If the force R be represented by the length $\overline{AB}$ laid off from its point of application (A) upon the line along which it acts, and if we draw through the point (A) two axes perpendicular to each other, we may substitute for the force R the two forces represented in intensity and

direction by the projections of the line AB upon the two axes.

Corollary II.

Let us conceive now, that two forces P, Q, being applied to the same point (A), and represented by two lines $\overline{AB}$, $\overline{AC}$, which form any angle between them, to be traced in the plane of these two forces, two axes, one of which coincides with the diagonal of the parallelogram to which they serve as sides, and the other perpendicular to this diagonal. We can substitute for the two forces P, Q, the four forces represented in intensity and direction by the projections of the lines $\overline{AB}$, $\overline{AC}$ upon the two axes. Now, of these four forces, two, being directly opposed, will be in equilibrium. The other two, directed along the diagonal of the parallelogram, will be added together, and will give, for their sum, a force represented in intensity and direction by this same diagonal. Hence we may enunciate the following proposition:

THEOREM.

The resultant R *of two forces* P, Q *simultaneously applied to a material point* (A) *and directed in any manner whatever, is represented in intensity and direction by the diagonal of the parallelogram constructed upon these two forces.*

Corollary I.

As the diagonal R of the parallelogram constructed upon the two forces P, Q is at the same time the third side of the triangle, which is formed by drawing through the extremity of the first force a line equal and parallel to the second, and as the angle in this triangle, opposite to the side R, is the supplement of the angle $(\widehat{P, Q})$, we have necessarily, by virtue of a known formula in trigonometry,

$$R^2 = P^2 + Q^2 + 2PQ \cos(\widehat{P, Q}) \dots\dots \qquad (2).$$

Corollary II.

In the case where the forces P, Q become equal to each other, their resultant R is represented in intensity and direction by the diagonal of the lozenge constructed upon these same forces. Then the formula (2) reduces to

$$R^2 = 2P^2 \{1 + \cos(\widehat{P, Q})\}. \qquad (3)$$

Moreover, by supposing $(\widehat{P, R}) = \theta$, we will find, in the present case, $(\widehat{P, Q}) = \theta$; and as we have, in general,

$$\cos 2\theta = 2 \cos^2 \theta - 1, \qquad (4)$$

the equation (3) will give $R = 2P \cos \theta$, or, what is the same,

$$R = 2P \cos(\widehat{P, R}). \qquad (5)$$

For the rest, we may be directly assured that the second member of the formula (5) represents the diagonal of the lozenge constructed upon the two forces equal to P.

It is easy to demonstrate the theorem of page 211, for the case in which the forces P, Q have any ratio to each other, when we have once established this theorem for the case where we have Q=P, that is to say, when formula (5) is established. Now, we can give a direct demonstration of this formula, which, in fact, is deduced from equation (4), but which appears to merit special remark. I will explain it in a few words.

Let us admit that formula (5) is verified for the case where we have $(\widehat{P, R})=\tau$, π designating either a right angle or an acute angle. I say that it will still subsist, if we suppose

$$(\widehat{P, R})=\frac{\tau}{2}, \text{ or } (\widehat{P, R})=\frac{\pi}{2}-\frac{\tau}{2}.$$

Now, in these two hypotheses, the angle $(\widehat{P, Q})$, included between the directions of the two equal forces P, Q, will be equivalent to one of the angles τ, $\pi-\tau$; and we can prove by reasoning as in Lemma II, that we can substitute in the system of the two forces P, Q, or for their resultant R, four components equal to $\frac{P^2}{R}$, among which, two will be directed along the same line and in the same direction as the force R, while the other two will each form with the resultant R an angle equivalent to $(\widehat{P, R})$, that is to say, to τ or to $\pi-\tau$. Now, since we

suppose formula (5) to be verified in the case where we have $(\widehat{P, R})=\tau$, the last two components evidently can be replaced by a single force equal to $2\frac{P^2}{R}\cos\tau$, and situated in the direction of the resultant R, or in the opposite direction. Consequently, we will find definitively,

$$R=2\frac{P^2}{R}\pm 2\frac{P^2}{R}\cos\tau=2\frac{P^2}{R}(1\pm\cos\tau);$$

or,

$$R^2=2P^2(1\pm\cos\tau), \qquad (6)$$

the sign $\pm$ being necessarily reduced to the sign $+$, in the case where $(\widehat{P, R})=\frac{\tau}{2}$, and to the sign $-$ in the case where $(\widehat{P, R})=\frac{\pi}{2}-\frac{\tau}{2}$. Since we will also get the formula (4) by placing successively

$$\theta=\frac{\tau}{2} \text{ and } \theta=\frac{\pi}{2}-\frac{\tau}{2},$$

$$1+\cos\tau=2\cos^2\frac{\tau}{2},\ 1-\cos\tau=2\cos^2\frac{\pi-\tau}{2},$$

equation (6) will give in the first case

$$R=2\cos\frac{\tau}{2},$$

and, in the second,

$$R=\cos\frac{\pi-\tau}{2}.$$

Hence, if equation (4) subsists when we attribute to the angle $(\widehat{P, R})$ the value τ, it will also subsist when we

attribute to the same angle one of the values $\frac{\tau}{2}$, $\frac{\pi-\tau}{2}$.

Now, this equation is verified when we suppose $\widehat{(\text{P}, \text{R})}=\frac{\pi}{2}$, since we have evidently in this hypothesis $\text{R}=0$, and $\cos\widehat{(\text{P}, \text{R})}=0$. Hence, it will be equally true if we suppose

$$\widehat{(\text{P}, \text{R})}=\frac{1}{2}\cdot\frac{\pi}{2}=\frac{\pi}{4},$$

and, consequently, if we take

$$\widehat{(\text{P}, \text{R})}=\frac{1}{2}\cdot\frac{\pi}{4}=\frac{\pi}{8}, \text{ or } \widehat{(\text{P}, \text{R})}=\frac{1}{2}\left(\pi-\frac{\pi}{4}\right)=\frac{3\pi}{8}.$$

Hence it will also be true, if we attribute to the angle $\widehat{(\text{P}, \text{R})}$ one of the values,

$$\frac{1}{2}\cdot\frac{\pi}{8}=\frac{\pi}{16}, \frac{1}{2}\cdot\frac{3\pi}{8}=\frac{3\pi}{16}, \frac{1}{2}\left(\pi-\frac{3\pi}{8}\right)=\frac{5\pi}{16},$$

$$\frac{1}{2}\left(\pi=\frac{\pi}{16}\right)=\frac{7\pi}{16}, \ldots$$

By continuing in the same manner, it may be proved that formula (5) generally has place when the angle $\widehat{(\text{P}, \text{R})}$ receives a value of the form $\frac{2m+1}{2}$, n representing any whole number, and $2m+1$ an odd number less than 2^n. If we now represent by θ an acute angle taken at will, we can vary the whole numbers m and n in such a manner that the ratio $\frac{2m+1}{2^n}$ will indefinitely approach the

limit θ; and the resultant R will tend more and more to coincide, on the one hand, with a force equivalent to $2P \cos\theta$, and, on the other hand, with the resultant of two forces equal to P, which would form between them an angle double of θ. Hence, this last resultant will be represented in magnitude by $2P \cos\theta$, and will also verify formula (5).

THE END.

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